AN INTRODUCTION TO REAL CLIFFORD ALGEBRAS AND THEIR CLASSIFICATION

by

Christopher S. Neilson
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Committee:

___	Dr. David Singman, Thesis Director
___	Dr. Rebecca F. Goldin, Committee Member
___	Dr. Jay A. Shapiro, Committee Member
___	Dr. Stephen Saperstone, Department Chairperson
___	Dr. Timothy L. Born, Associate Dean for Student and Academic Affairs, College of Science
___	Dr. Vikas Chandhoke, Dean, College of Science
Date: July 27, 2012	Summer Semester 2012 George Mason University Fairfax, VA

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By

Christopher S. Neilson
Master of Science
University of Florida, 2002
Bachelor of Science
The College of William & Mary, 2000

Director: Dr. David Singman, Professor
Department of Mathematical Sciences

Summer Semester 2012
George Mason University
Fairfax, VA

Dedication

To Genni,

Here's to finishing what we start.

Acknowledgments

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$\bigoplus$	direct sum, page 7
$\bigotimes$	tensor product, page 39
$\bigwedge(V)$	exterior algebra of the finite-dimensional, real vector space V, page 48
$\mathcal{C}\ell(V,Q)$	Clifford algebra of the finite-dimensional, real vector space V with nondegenerate quadratic form Q, page 58
$\mathcal{C}\ell_{p,m}$	Clifford algebra of a $(p+m)$-dimensional, real vector space having quadratic form with signature $(p,m,0)$, page 71
$\cong$	"is isomorphic to"
$\mathbb{C}$	complex numbers
$\mathbb{C}(n)$	algebra of $n \times n$ matrices with complex entries over the field of reals, page 81
$\mathbb{F}$	field of either real or complex numbers, page 2
$\mathbb{H}$	quaternions, page 73
$\mathbb{H}(n)$	algebra of $n \times n$ matrices with quaternion entries over the field of reals
$\mathbb{K}$	real numbers, complex numbers, or quaternions, page 79
$\mathbb{K}(n)$	algebra of n-by-n matrices with entries from the real numbers, complex numbers or quaternions, page 79
$\mathbb{R}$	real numbers
$\mathbb{R}(m,n)$	m-by-n matrices with entries from the field of real numbers, page 5
$\mathbb{R}(n)$	algebra of n-by-n matrices with entries from the field of real numbers, page 12
∇	gradient operator
$\otimes_{\mathbb{R}}$	tensor product over the field of real numbers
$\mathbb{R}[x]$	ring of polynomials over $\mathbb{R}$ of the single indeterminate x, page 60
$\sim$	equivalence relation, page 1
$\wedge$	exterior product (wedge product), page 53
C^1	set of once differentiable real functions

C^∞	set of infinitely differentiable real functions
$N(\sigma)$	number of inversions associated with a permutation σ, page 21

Abstract

AN INTRODUCTION TO REAL CLIFFORD ALGEBRAS AND THEIR CLASSIFICATION

Christopher S. Neilson, M.S.

George Mason University, 2012

Thesis Director: Dr. David Singman

Real Clifford algebras are associative, unital algebras that arise from a pairing of a finite-dimensional real vector space and an associated nondegenerate quadratic form. Herein, all the necessary mathematical background is provided in order to develop some of the theory of real Clifford algebras. This includes the idea of a universal property, the tensor algebra, the exterior algebra, and $\mathbb{Z}_2$-graded algebras. Clifford algebras are defined by means of a universal property and shown to be realizable algebras that are nontrivial. The proof of the latter fact is fairly involved and all details of proof are given. A method for creating a basis of any Clifford algebra is given. We conclude by giving a classification of all real Clifford algebras as various matrix algebras.

Part I

Preliminaries

Chapter 1: Basic Algebraic Concepts

This chapter is a collection of miscellaneous concepts from abstract algebra that will be necessary background for the topics covered in the main presentation of this work. These concepts can be found in many texts on abstract algebra or linear algebra such as [DF04] or [Rom08].

1.1 Equivalence Relations

Definition 1.1. A relation $\sim$ between elements in a nonempty set S is an **equivalence relation** if, for all $x, y, z \in S$, the following conditions are met:

1. $x \sim x$ (reflexive property),
2. if $x \sim y$, then $y \sim x$ (symmetric property), and
3. if $x \sim y$ and $y \sim z$, then $x \sim z$ (transitive property).

The three defining properties of equivalence relations are those we associate with $=$, the relation of equality. Equivalence relations are a generalization of equality; loosely speaking, equivalence relations provide alternate ways one may view elements of a set as being equivalent or "equal".

Given an element a in a nonempty set S with an equivalence relation $\sim$, the set $[a] = \{x \in S \mid x \sim a\}$ is called an *equivalence class*. If follows from the reflexive property that each element of S is in at least one equivalence class. It follows from the symmetric and transitive properties that each element is in at most one equivalence class. Therefore, every element of S is in exactly one equivalence class; the union of all the equivalence classes of $\sim$ equals S; and any two distinct equivalence classes are disjoint. Any element of an equivalence class is said to be a *representative* of the equivalence class.

1.2 Vector Spaces

Definition 1.2. Let V be a nonempty set with two binary operations defined upon it: *vector addition* $+ : V \times V \to V$ and *scalar multiplication* $\cdot : \mathbb{F} \times V \to V$, where $\mathbb{F}$ is either the field of reals or complex numbers. The collection $(V, +, \cdot)$ is a **vector space** if it meets the following criteria for all $x, y, z \in V$ and any $r, s, t \in \mathbb{F}$:

1. (additive associativity) $(x + y) + z = x + (y + z) = x + y + z\,;$
2. (multiplicative associativity) $r \cdot (s \cdot x) = (rs) \cdot x = rsx\,,$ note the $\cdot$ operator will normally be omitted and scalar multiplication will be denoted by juxtaposition;
3. (additive commutativity) $x + y = y + x\,;$
4. (distributivity) $r(x + y) = rx + ry$ and $(r + s)x = rx + sx\,;$
5. (existence of an additive identity) there exists a unique *zero vector* denoted by 0 such that $x + 0 = x$ for all $x \in V\,;$
6. (existence of a multiplicative identity) $1x = x\,;$
7. (existence of additive inverses) for each x there exists a unique element $-x$ such that $x + (-x) = 0\,.$

When referring to a vector space $(V, +, \cdot)$, typically the binary operations are omitted and the vector space is called simply V. The elements of a vector space are called *vectors* and the elements of $\mathbb{F}$ are called *scalars*. When field $\mathbb{F}$ is taken to be the real numbers, V will be called a real vector space; when $\mathbb{F}$ is the complex numbers, V is called a complex vector space. Alternatively, a vector space V with scalars in $\mathbb{F}$ may be referred to as a vector space over $\mathbb{F}$.

In this treatise, we will deal almost exclusively with real vector spaces. The exception is in Chapter 6, where complex vector spaces will be used in the proof of Proposition 6.7.

Definition 1.3. Given any finite collection of vectors $v_1, v_2, \ldots, v_n$ in a vector space V, and any scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$ in $\mathbb{F}$, a **linear combination** of those vectors is the sum

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n \,.$$

Definition 1.4. Let S be a set of vectors in V. If, for any finite collection of vectors $v_1, v_2, \ldots, v_n \in S$, the condition $\alpha_1 v_1 + \cdots + \alpha_n v_n = 0$ implies that $\alpha_1 = \cdots = \alpha_n = 0$, then S is said to be **linearly independent**. If S is not linearly independent, then it is said to be **linearly dependent**.

Definition 1.5. Let S be a set of vectors in V. The set $W = \{\alpha_1 v_1 + \cdots + \alpha_n v_n \mid n \in \mathbb{N}\,;\; v_1, \ldots, v_n \in S \text{ and } \alpha_1, \ldots, \alpha_n \in \mathbb{F}\}$, is called the **span** of S; also, S is said to **span** W.

Definition 1.6. A **basis** for a vector space V is a linearly independent collection of vectors that spans V.

A standard argument using Zorn's Lemma guarantees that every vector space has a basis. A vector space is said to be *finitely generated* if it has a basis set that is finite.

Proposition 1.1. Given a basis of a finite dimensional vector space, any vector in the space can be expressed uniquely as a linear combination of the basis elements.

Proof. Let $\{b_1, \ldots, b_n\}$ be a basis for vector space V. For any vector $v \in V$, suppose there are two linear combinations of the basis elements that equal v. Then, there exists two sets of scalars, $\{\alpha_1, \ldots, \alpha_n\}$ and $\{\beta_1, \ldots, \beta_n\}$ such that

$$v = \alpha_1 b_1 + \cdots + \alpha_n b_n = \beta_1 b_1 + \cdots + \beta_n b_n \,.$$

From this it is evident that

$$(\alpha_1 - \beta_1) b_1 + \cdots + (\alpha_n - \beta_n) b_n = 0 \,.$$

Since the b_i are linearly independent, this means that for each $i \in \{1, \ldots, n\}$ the coefficient $(\alpha_i - \beta_i) = 0$. Therefore, $\alpha_i = \beta_i$, that is, the two linear combinations are in fact the same. $\square$

Proposition 1.2. If $S = \{v_1, v_2, \ldots, v_n\}$ is a spanning set for a vector space V, then any collection of m vectors in V, where $m > n$, is linearly dependent.

Proof. Take any finite collection of vectors $\{u_1, u_2, \ldots, u_m\}$ in V such that $m > n$. Each u_i can be written as

$$u_i = \sum_{j=1}^{n} \alpha_{ij} v_j$$

since S spans V. Consider a linear combination of the u_i set equal to zero: $\beta_1 u_1 + \beta_2 u_2 + \cdots + \beta_m u_m = 0$. By definition, if a nontrivial solution exists (i.e., a solution in which not all the β_i are equal to zero), then the u_i are linearly dependent. We proceed by writing the linear combination in terms of the v_j.

$$\begin{aligned} 0 = \sum_{i=1}^{m} \beta_i u_i &= \sum_{i=1}^{m} \beta_i \left(\sum_{j=1}^{n} \alpha_{ij} v_j \right) \\ &= \sum_{j=1}^{n} \left(\sum_{i=1}^{m} \beta_i \alpha_{ij} \right) v_j \end{aligned}$$

This equation, now in terms of v_j, always has at least the trivial solution in which each coefficient is equal to zero:

$$\sum_{i=1}^{m} \beta_i \alpha_{ij} = 0 \quad \text{for } j = 1, 2, \ldots, n. \tag{1.1}$$

We would like to understand in more detail what values the β_i can take so we consider them as variables. The α_{ij}, on the other hand, are fixed and we assume they are known.

Equation 1.1 then gives a homogeneous system of n linear equations in m variables. There are more variables than equations ($m > n$) so the system always has a nontrivial solution. Therefore, the vectors $u_1, u_2, \ldots, u_m$ are linearly dependent. □

Corollary 1.3. For any finitely generated vector space V, any two bases of V have the same cardinality.

Proof. Suppose that $B_1 = \{b_1, \ldots, b_m\}$ and $B_2 = \{e_1, \ldots, e_n\}$ are two bases of a vector space V. Basis B_1 spans V and B_2 is a linearly independent set in V, so $n \leq m$ by Proposition 1.2. However, the same is true if we reverse the roles of B_1 and B_2, that is, B_2 spans V and B_1 is a linearly independent set, so $m \leq n$. Therefore, $m = n$. □

Having shown that any two bases of a finitely generated vector space contain the same number of vectors, it is now possible to unambiguously categorize such a vector space according to the number of vectors in a basis. This number is called the *dimension* of the vector space.

Definition 1.7. The **dimension** of a vector space V, denoted $\dim V$, is equal to the cardinality of any basis for V.

Example 1.1. Taking n to be a positive integer, the n-fold Cartesian product $\prod_{i=1}^{n} \mathbb{F}$ can be made a vector space with appropriate definitions of vector addition and scalar multiplication. The elements of the Cartesian product are n-tuples; for two arbitrary n-tuples, $v = (v_1, \ldots, v_n)$ and $u = (u_1, \ldots, u_n)$, vector addition is defined by adding corresponding components: $v + u = (v_1 + u_1, \ldots, v_n + u_n)$. Scalars are elements of $\mathbb{F}$ and scalar multiplication is defined for all $\alpha \in \mathbb{F}$ by $\alpha \cdot v = (\alpha v_1, \ldots, \alpha v_n)$.

Example 1.2. Let $\mathbb{R}(m, n)$ denote the set of all m-by-n matrices with entries from the field of real numbers. Define scalar multiplication on this set such that for any $c \in \mathbb{R}$ and any $A \in \mathbb{R}(m,n)$, where

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix},$$

the product cA is given by

$$cA = \begin{pmatrix} ca_{11} & ca_{12} & \cdots & ca_{1n} \\ ca_{21} & ca_{22} & \cdots & ca_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ ca_{m1} & ca_{m2} & \cdots & ca_{mn} \end{pmatrix}.$$

This scalar multiplication, along with the standard definition of componentwise matrix addition, make $\mathbb{R}(m,n)$ a real vector space. If E_{ij} is the matrix consisting of all zeros except for a 1 in the ith row and jth column, then the set $B = \{E_{ij} \mid 1 \le i \le m \text{ and } 1 \le j \le n\}$ is a basis for $\mathbb{R}(m,n)$. There are mn basis vectors E_{ij} so $\mathbb{R}(m,n)$ has dimension mn.

So far our discussion of vector spaces has primarily covered fundamental relationships between vectors within a vector space. Next we introduce the direct sum, one method for creating a new vector space from existing vector spaces. Before doing so, we note that a function f, defined on a set S, is said to have *finite support* if $f = 0$ everywhere on S except for a finite subset of S on which f is nonzero.

Definition 1.8 (Direct sum of real vector spaces)**.** Let $\mathcal{V} = \{V_i \mid i \in I\}$ be a collection of real vector spaces indexed by the set I. Let

$$\mathscr{F} = \{f : I \to \bigcup_{i \in I} V_i \mid f(i) \in V_i \text{ for each } i \in I\}$$

be the set of functions that map an element $i \in I$ into the vector space V_i. The subset of $\mathscr{F}$ consisting of only those f which have finite support is denoted by

$$\bigoplus_{i \in I} V_i = \{f : I \to \bigcup_{i \in I} V_i \mid f(i) \in V_i \text{ and } f \text{ has finite support}\}$$

and is called the **direct sum** of the V_i.

With the standard definitions of function addition and scalar multiplication, i.e.,

$$(f+g)(i) := f(i) + g(i) \quad \text{and}$$

$$(af)(i) := af(i)\,,$$

it follows from the vector space properties of each V_i that for all $i \in I$, the direct sum $\bigoplus_{i \in I} V_i$ is a vector space. The following are two examples of the direct sum, one general and one specific, where the index set is finite.

Example 1.3 (Finite direct sum (general))**.** The direct sum of the real vector spaces $V_1, \ldots, V_n$, where $n \in \mathbb{N}$, is denoted by

$$\bigoplus_{i=1}^{n} V_i = V_1 \oplus V_2 \oplus \cdots \oplus V_n\,.$$

A typical element of $V_1 \oplus \cdots \oplus V_n$ is given by the n-tuple $(v_1, v_2, \ldots, v_n)$ where each $v_i \in V_i$. Note that the n-tuples are nothing more than mappings from the index set $\{1, 2, \ldots, n\}$ to $\bigcup_{i=1}^{n} V_i$, so expressing the elements of our direct sum as n-tuples is equivalent to expressing them as functions as in Definition 1.8. Addition of elements is component-wise, so

$$(v_1, v_2, \ldots, v_n) + (w_1, w_2, \ldots, w_n) = (v_1 + w_1, v_2 + w_2, \ldots, v_n + w_n)$$

and for a real number r

$$r \cdot (v_1, v_2, \ldots, v_n) = (rv_1, rv_2, \ldots, rv_n)\,.$$

It can be shown that the dimension of a direct sum is the sum of the dimensions of its summands.

Example 1.4 (Direct sum (specific))**.** The direct sum of n copies of $\mathbb{R}$, that is, $\bigoplus_{i=1}^{n} \mathbb{R}$ is vector space isomorphic to $\mathbb{R}^n$, where the latter has the standard vector space structure described in Example 1.1.

The next theorem shows how to use direct sums in order to generate a vector space with a prescribed basis.

Theorem 1.4. Any set S is the basis for some vector space V_S.

Proof. Let $S = \{s_\alpha\}_{\alpha \in A}$, where A is an index set for S. For each α, let $V_\alpha = \{rs_\alpha \mid r \in \mathbb{F}\}$ be the set of objects rs_α for all $r \in \mathbb{F}$. Define the operation of vector addition $+ : V_\alpha \times V_\alpha \to V_\alpha$ such that for all $r, t, u \in \mathbb{F}$,

1. $rs_\alpha + ts_\alpha = (r + t)s_\alpha\,,$ and
2. $rs_\alpha + (ts_\alpha + us_\alpha) = (rs_\alpha + ts_\alpha) + us_\alpha \quad$ (associativity).

Next define the operation of scalar multiplication $\cdot : \mathbb{F} \times V_\alpha \to V_\alpha$ so that for all $r, t, u \in \mathbb{F}$,

1. $r \cdot ts_\alpha = (rt)s_\alpha = rts_\alpha\,,$
2. $r \cdot (ts_\alpha + us_\alpha) = r \cdot ts_\alpha + r \cdot us_\alpha = rts_\alpha + rus_\alpha\,,$ and
3. $(r + t) \cdot us_\alpha = r \cdot us_\alpha + t \cdot us_\alpha = rus_\alpha + tus_\alpha\,.$

Then each V_α is a one dimensional vector space over $\mathbb{F}$ and the direct sum of them, $V_S =$

$\bigoplus_{\alpha \in A} V_\alpha$ is a vector space over $\mathbb{F}$. For each $\alpha \in A$, let $f_\alpha : A \to V_S$ be defined by

$$f_\alpha(\gamma) = \begin{cases} 0 & \text{if } \gamma \neq \alpha\,, \\ s_\alpha & \text{if } \gamma = \alpha\,. \end{cases}$$

By identifying the function f_α, which has s_α in the α^{th} coordinate, with s_α itself, the set S is seen to be a basis for V_S. □

1.3 Linear and Multilinear Maps

Definition 1.9. Let V and W both be vector spaces over $\mathbb{F}$. A **linear function** or **linear map** $f : V \to W$ between vector spaces V and W is a function that satisfies the following for all $v_1, v_2 \in V$ and $\alpha, \beta \in \mathbb{F}$:

$$f(\alpha v_1 + \beta v_2) = \alpha f(v_1) + \beta f(v_2)\,.$$

Theorem 1.5 (Linear Extension)**.** Let V and W be finite-dimensional vector spaces over $\mathbb{F}$. Let $B = \{b_1, \ldots, b_n\}$ be a basis for V and define a map $T' : B \to W$ on each of the basis vectors. Then there is a unique linear map $T : V \to W$ such that $T\big|_B = T'$.

Proof. Any vector $v \in V$ can be uniquely represented as a linear combination of the basis vectors in B (Proposition 1.1). Thus, there exists a unique collection of scalars $\{\alpha_i\}_{i=1}^n$ such that $v = \sum_{i=1}^n \alpha_i b_i$. Therefore, T is defined on v as shown:

$$T(v) = T\left(\sum_{i=1}^n \alpha_i b_i\right)$$

$$T(v) = \sum_{i=1}^n \alpha_i T(b_i) = \sum_{i=1}^n \alpha_i T'(b_i)\,.$$

The map T is obviously unique since any other linear map L with the property $L\big|_B = T'$ will yield $L(v) = \sum_{i=1}^n \alpha_i T'(b_i) = T(v)$. □

Theorem 1.5 highlights an important property of linear maps which is that a linear map only requires its values on a basis to be specified in order to define the entire map. Implicit use of Theorem 1.5 will be made repeatedly throughout this thesis for the purpose of defining linear maps. In practice, when defining a linear map $T : V \to W$ in this manner, we dispense with first defining the intermediate map T' as was done in the theorem. Instead, we define T explicitly on a basis and indicate that this is meant to define T on all of V by using terminology such as *linear extension* of T or *linearly extending* T to all of V.

Definition 1.10. A **vector space isomorphism** is a linear, bijective map between vector spaces.

If two vector spaces possess an isomorphism between them, they are said to be *isomorphic.* If a vector space isomorphism $T : V \to V$ maps a vector space V into itself, then T is referred to as a *vector space automorphism.*

Definition 1.11. Let $V_1, V_2, \ldots, V_n$ and V each be vector spaces over $\mathbb{F}$. Let $u_i, v_i \in V_i$ be vectors from the vector spaces with the corresponding index, and let $\alpha, \beta \in \mathbb{F}$ be scalars. A map $f : V_1 \times \cdots \times V_n \to V$ is a **multilinear function** if it has the following property:

$$f(v_1, \ldots, \alpha u_j + \beta v_j, \ldots, v_n) = \alpha f(v_1, \ldots, u_j, \ldots v_n) + \beta f(v_1, \ldots, v_j, \ldots v_n)$$

for each $j = 1, \ldots, n$, each $u_j, v_j \in V_j$ and each $\alpha, \beta \in \mathbb{F}$.

In the same way that linear maps can be uniquely defined by specifying their values on a basis, multilinear maps can also be defined on a much smaller set and extended uniquely. The following theorem on multilinear extension is proved in a similar fashion to Theorem 1.5.

Theorem 1.6 (Multilinear Extension)**.** Let $V_1, \cdots, V_n$, and W be a finite collection of vector spaces over $\mathbb{F}$. Let B_i be a basis for V_i and $\mathcal{B} = B_1 \times \cdots \times B_n$. Define $T' : \mathcal{B} \to W$

arbitrarily. Then there is a unique multilinear map $T : V_1 \times \cdots \times V_n \to W$ such that $T\big|_{\mathcal{B}} = T'$.

1.4 Algebras

Definition 1.12. A **real algebra** $(A, +, \cdot, *)$ is a nonempty set A, along with the three operations of addition $(+)$, scalar multiplication $(\cdot)$ by elements from the field of real numbers $\mathbb{R}$, and multiplication $(*)$ between elements of A, that have the following properties for all $a, b, c \in A$ and $r \in \mathbb{R}$:

1. $(A, +, \cdot)$ is a real vector space;

2. A is closed with respect to $*$ (closure with respect to $+$ and $\cdot$ follows from property 1);

3. multiplication is associative, $(a * b) * c = a * (b * c)$;

4. multiplication distributes over addition, i.e., $a * (b + c) = a * b + a * c$ and $(a + b) * c = a * c + b * c$; and

5. $r \cdot (a * b) = (r \cdot a) * b = a * (r \cdot b)$.

Thus, an algebra can be thought of as a vector space in which the vectors can be multiplied together [Rom08]. Often other authors do not require that algebras be associative with respect to multiplication, however all the algebras considered herein are associative. To reduce repetition as we proceed, the associativity requirement is included up front in our definition. It is not required that an algebra contain a multiplicative identity, however, if an algebra does contain such an element then the algebra is called a *unital algebra*. We will adopt the convention of using juxtaposition to denote both the algebra multiplication and the scalar multiplication whenever no confusion will result.

Example 1.5 (Matrix algebra)**.** Let $\mathbb{R}(m) = \mathbb{R}(m, m)$, the real vector space defined in Example 1.2. Defining the standard matrix multiplication on $\mathbb{R}(m)$ yields a real unital algebra.

Definition 1.13. Let B be a subset of the real algebra $(A, +, \cdot, *)$. Set B is said to be a **subalgebra** of A if it is closed with respect to the binary operators on A. That is, for all $x, y \in B$ and $r \in \mathbb{R}$,

1. $x + y \in B$,
2. $xy \in B$, and
3. $rx \in B$.

If these three conditions are met, the elements of B inherit all the properties of an algebra due to their being elements of algebra A.

Definition 1.14 (generating set of an algebra)**.** Let S be a nonempty subset of the real algebra A and let B be the intersection of all subalgebras of A that contain S. Then S is a **generating set** of the subalgebra B and B is said to be *generated by* S.

Since B is the result of an intersection, it is said to be the "smallest" subalgebra containing S. Note that this intersection is never empty since A is itself a subalgebra containing S. If $B = A$, then S is said to be the generating set of the algebra A. The elements of S are called the *generators* of B.

Definition 1.14 is tidy, but doesn't necessarily offer insight into the elements in a generating set, or conversely, given a generating set S with a multiplication rule, what algebra it produces. Since S is contained in an algebra, and an algebra is closed, this means that the algebra generated by S consists of all elements that have the form

$$\sum_{i=1}^{n} \alpha_i \left(\prod_{j=1}^{k_i} s_j \right),$$

where $s_j \in S$ and $n, k_i \in \mathbb{N}$. That is, the algebra generated by S consists of all possible linear combinations of finite products of elements of S. Note that for a given term in the linear combination, the s_j are not required to be unique.

Definition 1.15. An algebra $\mathcal{A}$ over the reals is called a **graded algebra** if it can be written as a direct sum of the form

$$\mathcal{A} = \bigoplus_{n \in \mathbb{N}} A_n$$

where the A_n are real vector spaces that are subspaces of $\mathcal{A}$, and that if $a_i \in A_i$ and $a_j \in A_j$ then

$$a_i a_j \in A_{i+j}\,.$$

1.4.1 Maps Between Algebras

Definition 1.16. An **algebra homomorphism** $\phi : A \to B$ is a map between real algebras A and B that has these properties for all $a_1, a_2 \in A$ and $r_1, r_2 \in \mathbb{R}$:

1. it is linear, i.e., $\phi(r_1 a_1 + r_2 a_2) = r_1 \phi(a_1) + r_2 \phi(a_2)\,;$
2. $\phi(a_1 a_2) = \phi(a_1)\phi(a_2);$ and
3. if there exists a multiplicative identity $1_A \in A$, then $\phi(1_A)$ is the multiplicative identity in B.

Definition 1.17. An **algebra isomorphism** is a bijective algebra homomorphism.

An algebra isomorphism that maps an algebra A into itself is referred to as an *algebra automorphism*.

1.4.2 Ideals

Definition 1.18. An **ideal** I is a subset of an algebra A such that I is a vector space with the added property that for all $x \in I$ and for all $a \in A$ the products xa and ax are elements of I.

An ideal $I_S \subset A$ is *generated* by a *generating set* S if I_S is the smallest ideal containing S. The intersection of any collection of ideals in A is also an ideal. In light of this fact, it is seen that I_S results from the intersection of all ideals containing S. This intersection is always nonempty since A is an ideal itself. The ideal I_S generated by S consists of elements of the following form

$$a_1 s_1 a_1' + a_2 s_2 a_2' + \cdots + a_n s_n a_n'$$

where $a_i, a_i' \in A$ and $s_i \in S$. To indicate I_S is generated by the set S, we write $I_S = \langle s \mid s \in S \rangle$.

1.4.3 Quotient Algebras

Given an algebra and an ideal it contains, it is possible to create a new algebra called a *quotient algebra*. In this section we describe the process of forming a quotient algebra and examine the elements it contains. We develop this structure because in Chapter 5 it will be seen that Clifford algebras are quotient algebras. The quotient algebra also makes an appearance in Section 4.4 with the introduction of the exterior algebra.

Given an algebra A containing an ideal I, define an equivalence relation, denoted by $\sim$, in the following way: for any $a, b \in A$, $a \sim b$ if and only if $a - b \in I$. A typical equivalence class $[b]$ resulting from this relation is the set

$$\begin{aligned}[b] &= \{a \in A \mid a \sim b\} \\ &= \{a \in A \mid a - b \in I\} \\ &= \{a \in A \mid a = b + x, \text{ for some } x \in I\} \\ &= \{b + x \mid x \in I\} .\end{aligned}$$

The set of all equivalence classes, $\{[a] \mid a \in A\}$, is denoted A/I and called "A modulo I" or simply "A mod I". It is a consequence of the definition of the equivalence relation that $[0] = I$, that is, the entire ideal is taken to be equivalent to 0 in A/I. Thus, use of the term "modulo" as an allusion to modular arithmetic of integers is appropriate as in that context "modulo n" makes the set of multiples of n equivalent to 0.

To imbue A/I with the structure of an algebra, the operations of addition, scalar multiplication, and the algebra multiplication $(+, \cdot, *)$ are defined in terms of these operations on algebra A; for any $a, b \in A$ and $r \in \mathbb{R}$,

$$\begin{aligned}[a] + [b] &= [a + b] , \\ r \cdot [a] &= [r \cdot a] , \quad \text{and} \\ [a] * [b] &= [a * b] .\end{aligned}$$

Any element in an equivalence class can be used to represent the class. This follows from the symmetric and transitive properties of equivalence relations which tell us that if $a \sim b$ then $[a] = [b]$. Therefore, in order for the addition and multiplication operations to be well-defined, they must hold for any element in an equivalence class. We proceed to demonstrate that these operations are indeed well-defined.

For addition, it must be shown that $a_1 \sim b_1$ and $a_2 \sim b_2$ implies that $a_1 + a_2 \sim b_1 + b_2$

because with this condition $[a_1]+[a_2] = [a_1+a_2] = [b_1+b_2] = [b_1]+[b_2]$. In fact, if $a_1-b_1 \in I$ and $a_2 - b_2 \in I$, then

$$I \ni (a_1 - b_1) + (a_2 - b_2) = a_1 + a_2 - (b_1 + b_2)$$

and so $a_1 + a_2 \sim b_1 + b_2$.

Scalar multiplication requires that $a \sim b$ implies $ra \sim rb$. If $a - b \in I$, then $ra - rb = r(a - b) \in I$, therefore $ra \sim rb$.

Finally, the algebra multiplication requires that $a_1a_2 \sim b_1b_2$ whenever $a_1 \sim b_1$ and $a_2 \sim b_2$. When $a_1 - b_1 \in I$ and $a_2 - b_2 \in I$, then

$$(a_1 - b_1)a_2 \in I \quad \text{and} \quad b_1(a_2 - b_2) \in I$$

and so

$$(a_1 - b_1)a_2 + b_1(a_2 - b_2) = a_1a_2 - b_1a_2 + b_1a_2 - b_1b_2 \in I\,,$$

yielding $a_1a_2 \sim b_1b_2$.

Given the quotient algebra A/I, there is a *canonical projection map* $\pi : A \to A/I$ defined for all $a \in A$ by $\pi(a) = [a]$. The projection map is surjective, since any element ϵ of A/I is a set which contains at least one element $a \in A$ and so $\epsilon = \pi(a)$. The projection map is also a homomorphism as $\pi(ab) = [ab] = [a][b] = \pi(a)\pi(b)$ for any $a, b \in A$.

Theorem 1.7. Let A and B be two real algebras, I an ideal of A, and $\pi : A \to A/I$ the projection mapping from A to the quotient space A/I. Let $T : A \to B$ be an algebra homomorphism. There exists a unique algebra homomorphism $\tau : A/I \to B$ such that $\tau \circ \pi = T$ if and only if $I \subseteq \ker T$.

Proof. Assume that τ exists. For any $x \in I$,

$$T(x) = (\tau \circ \pi)(x) = \tau(0 + I) = 0\,.$$

Thus, $I \subseteq \ker T$.

Next, assume $I \subseteq \ker T$. Define τ as stated in the theorem: $\tau(x+I) = (\tau\circ\pi)(x) = T(x)$. It must be shown that τ is well-defined and unique. For $x, y \in A$, suppose that $x+I = y+I$. Then,

$$x + I = y + I \quad \Rightarrow \quad x - y \in I \quad \Rightarrow \quad T(x - y) = 0 \quad \Rightarrow \quad T(x) = T(y)\,.$$

Therefore, $\tau(x + I) = (\tau \circ \pi)(x) = T(x) = T(y) = (\tau \circ \pi)(y) = \tau(y + I)$, indicating that τ is well-defined. To show uniqueness, assume there also exists a function τ' such that $\tau' \circ \pi = T$. For any $x \in A$,

$$(\tau' \circ \pi)(x) = T(x) = (\tau \circ \pi)(x) \quad \Rightarrow \quad \tau'(x + I) = \tau(x + I) \quad \Rightarrow \quad \tau' = \tau\,.$$

Therefore, τ is unique. □

Chapter 2: Permutations

Definition 2.1. Let S be a nonempty finite set. A **permutation** of S is a bijective function from S into S.

For simplicity, in this section we let S be $\{1, 2, \ldots, n\}$ for some n. Given two permutations σ_1 and σ_2, their product $\sigma_1\sigma_2$ is simply the function composition $\sigma_1 \circ \sigma_2$ of the two permutations. Note that here, σ_2 is applied first. This leads to a very natural notation in which, for example, $\sigma \circ \sigma$ is denoted σ^2 and $\sigma^{-1} \circ \sigma^{-1} \circ \sigma^{-1}$ is denoted σ^{-3}. In this notation, $\sigma^0 = \sigma\sigma^{-1}$ is the identity.

There are multiple notations to denote a permutation. Two of these notations will be given here. The first, often called the *row notation*, is illustrated in the following example.

Example 2.1 (Row notation for permutations)**.** Let $S = \{1, 2, 3, 4, 5\}$. The permutation σ on S given by $\sigma(1) = 2$, $\sigma(2) = 4$, $\sigma(3) = 1$, $\sigma(4) = 3$, and $\sigma(5) = 5$ is denoted as

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 1 & 3 & 5 \end{pmatrix}.$$

The second notation uses *cycles*.

Definition 2.2. A **cycle** is a permutation μ on a set S such that there exists a subset $A = \{a_1, a_2, \ldots, a_n\}$ of S that defines μ in the following way

$$\mu(a_i) = \begin{cases} a_{i+1} & \text{for } 1 \le i \le n-1\,, \\ a_1 & \text{for } i = n \end{cases}$$

and

$$\mu(x) = x \quad \text{for all } x \notin A\,.$$

Therefore, the permuted elements of A of a cycle μ on S can be obtained from repeated application of μ to any $a_i \in A$; all other elements in S are constant under μ.

A cycle is represented by writing down the elements it permutes and omitting the elements it holds fixed, as follows:

$$\begin{pmatrix} a_1 & a_2 & \dots & a_n \end{pmatrix}\,.$$

Writing the permutation from Example 2.1 in cycle notation yields:

$$\sigma = \begin{pmatrix} 1 & 2 & 4 & 3 \end{pmatrix}\,.$$

The number of integers that appear in a cycle is called the *length* of the cycle. Two cycles are said to be *disjoint* if elements permuted by one are all different from the elements permuted by the other. That is, if c_1 and c_2 are cycles on the set S and c_1 permutes the elements in $A_1 \subseteq S$ and c_2 permutes the elements in $A_2 \subseteq S$, then c_1 and c_2 are disjoint if $A_1 \cap A_2 = \varnothing$.

Theorem 2.1. Every permutation σ can be expressed as a product of disjoint cycles.

Proof. Let σ act on $S = \{1, 2, \dots, n\}$ and let $i, j, k \in S$. Define the equivalence relation $\sim$ by

$$i \sim j \Leftrightarrow \sigma^m(i) = j \quad \text{for some } m \in \mathbb{Z}\,.$$

The equivalence relation partitions the set S into the (disjoint) equivalence classes S_1,

$S_2, \ldots, S_r$ where $1 \le r \le n$. To each S_ℓ associate a permutation σ_ℓ such that

$$\sigma_\ell(i) = \begin{cases} \sigma(i) & \text{if } i \in S_\ell\,, \\ i & \text{if } i \notin S_\ell\,. \end{cases}$$

For an appropriate choice of labelling, it follows from the equivalence relation that if $S_\ell = \{a_1, a_2, \ldots, a_q\}$ then

$$\sigma_\ell(a_s) = \begin{cases} a_{s+1} & \text{if } 1 \le s \le q-1\,, \\ a_1 & \text{if } s = q\,. \end{cases}$$

Thus, σ_ℓ is a cycle. Finally,

$$\sigma = \sigma_1\,\sigma_2\,\cdots\,\sigma_r$$

since if $i \in S_\ell$ then $\sigma_\ell(i) = \sigma(i)$ but otherwise $\sigma_\ell(i) = i$. □

Writing a permutation σ as a product of disjoint cycles is called the *cycle decomposition* of σ.

Another important type of permutation is called a *transposition*.

Definition 2.3. A **transposition** is a permutation that interchanges two elements only. That is, suppose I indexes a set S and τ is a permutation on S. For $\alpha \in I$ and $s_\alpha \in S$, then τ is a transposition if there exists $\beta, \gamma \in I$ such that

$$\tau(s_\beta) = s_\gamma\,,$$

$$\tau(s_\gamma) = s_\beta\,, \quad \text{and}$$

$$\tau(s_\alpha) = s_\alpha \quad \text{whenever } \alpha \ne \beta \text{ and } \alpha \ne \gamma\,.$$

Thus, transpositions are cycles of length two. For example, the transposition $\tau : \{1,2,3,4\} \to \{1,2,3,4\}$, represented in cycle notation as $(1\ 3)$, sends $1 \mapsto 3$ and $3 \mapsto 1$ and keeps the elements 2 and 4 fixed.

Theorem 2.2. Every cycle can be decomposed into a product of transpositions.

Proof. Given any cycle $c = (a_1\ a_2\ \cdots\ a_n)$ of length n,

$$c = (a_1\ a_n)(a_1\ a_{n-1}) \cdots (a_1\ a_2)\,.$$

□

Corollary 2.3. Every permutation σ can be expressed as a product of transpositions.

Proof. The corollary follows immediately from Theorems 2.1 and 2.2. □

Definition 2.4. Given a permutation σ on a set S, an **inversion** is a pair of elements $i, j \in S$ for which $i < j$ and $\sigma(i) > \sigma(j)$.

Given all possible pairs of elements from S, let $N(\sigma)$ be the total number of pairs that are inversions. Using row notation, there is a simple method for counting the number of inversions of a permutation σ on $\{1, 2, \ldots, n\}$. Working with the second row, start with the number in the first slot (i.e., $\sigma(1)$) and in turn, examine each number to the right (i.e., $\sigma(2)$, $\sigma(3)$, $\ldots$, $\sigma(j)$, $\ldots$, $\sigma(n)$). If $\sigma(1) > \sigma(j)$ then the pair $(1, j)$ is an inversion since, obviously, $1 < j$. Next, start with $\sigma(2)$ and compare it to each $\sigma(j)$ to the right. Again, when $\sigma(2) > \sigma(j)$ the pair $(2, j)$ will be an inversion since we are only considering cases with $j > 2$. Continue the process through comparison of $\sigma(n-1)$ with $\sigma(n)$, at which point the number of inversions is found since all possible pairs (i, j) have been considered.

Definition 2.5. The **sign** or **parity** of a permutation σ is

$$\operatorname{sgn}(\sigma) = (-1)^{N(\sigma)}\,.$$

The permutation σ is called *even* if $\operatorname{sgn}(\sigma) = 1$ and it is called *odd* if $\operatorname{sgn}(\sigma) = -1$.

Lemma 2.4. Given a transposition $\tau = (i \;\; j)$ on the ordered set $S = \{1,\, 2,\, \ldots,\, n\}$, where $i < j$, the transposition can be factored into a product of transpositions of the form $\tau_\alpha = (i_\alpha \;\; j_\alpha)$, where i_α and j_α are adjacent in S. The product has an odd number of factors.

Proof. The transposition $\tau = (i \;\; j)$ can be decomposed into the product

$$\tau = \underbrace{(i \;\; i+1)(i+1 \;\; i+2)\cdots(j-1 \;\; j)}_{\zeta}\underbrace{(j-1 \;\; j-2)\cdots(i+1 \;\; i)}_{\xi}$$

where ζ and ξ are the permutations shown above. The permutation ζ is composed of $j - i$ transpositions and ξ is composed of $(j-1) - i$ transpositions. So, $\tau = \zeta\xi$ is composed of $j - i + (j-1) - i = 2(j-i) - 1$ transpositions, which is an odd number of transpositions. □

Theorem 2.5. Let σ be a permutation acting on the set $X = \{1, 2, \ldots, n\}$, where n is a positive integer and X has the usual ordering. If σ can be factored as a product of ℓ transpositions and a product of m transpositions, then ℓ and m must both be even or both be odd.

Proof. Let $\sigma = T_1T_2\cdots T_\ell$ be one decomposition of σ into transpositions and let $\sigma = Q_1Q_2\cdots Q_m$ be another. By Lemma 2.4 each T_k can be factored into a product of an odd number of transpositions, where each factor interchanges adjacent elements of X. The permutation σ, written as a product of these factors, is

$$\sigma = \tau_1\tau_2\cdots\tau_{\ell'}\,.$$

A similar decomposition of the Q_k gives a factorization

$$\sigma = \theta_1\theta_2\cdots\theta_{m'}\,.$$

If ℓ' is odd, then ℓ is odd (and likewise with m' and m). If ℓ' is even, then ℓ is even (and likewise with m' and m).

Now, consider $\tau_1^{-1}\sigma$. The transposition τ_1^{-1} interchanges some pair $(i, i+1)$ and so the number of inversion pairs of $\tau_1^{-1}\sigma$ differs from $N(\sigma)$ by one:

$$N(\tau_1^{-1}\sigma) = \begin{cases} N(\sigma) - 1 & \text{if } (i, i+1) \text{ is an inversion pair of } \sigma\,, \\ N(\sigma) + 1 & \text{if } (i, i+1) \text{ is not an inversion pair of } \sigma\,. \end{cases}$$

Next, repeat this procedure of composing τ_k^{-1} with $\tau_{k-1}^{-1} \cdots \tau_2^{-1}\tau_1^{-1}\sigma$ until we have

$$\tau_{\ell'}^{-1} \cdots \tau_2^{-1}\tau_1^{-1}\sigma = \tau_{\ell'}^{-1} \cdots \tau_2^{-1}\tau_1^{-1}\tau_1\tau_2 \cdots \tau_{\ell'} = \mathrm{Id}\,,$$

where Id is the identity permutation of X. The number of inversions of Id is $N(\mathrm{Id}) = 0 = N(\sigma) - p + q$ where p is the number of transpositions τ_k^{-1} that changed inversion pairs of σ to non-inversion pairs and q is the number of transpositions τ_k^{-1} that changed non-inversion pairs of σ to inversion pairs. Of course, $p + q = \ell'$ and so $N(\sigma) + 2q = \ell'$. A similar composition of the θ_k^{-1} with σ yields $N(\sigma) - r + s = 0$ and $r + s = m'$, which gives $N(\sigma) + 2s = m'$. Therefore, $\ell' - m' = 2q - 2s$, and so m' and ℓ' are either both odd or both even. $\square$

This theorem implies that the sign of a permutation can also be determined based on whether the permutation can be decomposed into an odd or even number of transpositions.

Corollary 2.6. Let $\sigma = \tau_1 \cdots \tau_n$ be a permutation factored as a product of n transpositions τ_j. The sign of σ is $\mathrm{sgn}(\sigma) = (-1)^n$.

Corollary 2.7. Given two permutations σ and μ, the sign of their product is is equal to the product of their signs, that is

$$\mathrm{sgn}(\sigma\mu) = \mathrm{sgn}(\sigma)\,\mathrm{sgn}(\mu)\,.$$

Proof. The permutation σ can be written as a product of n transpositions and μ can be written as a product of m transpositions. The product $\sigma\mu$ can therefore be written as a product of $n + m$ transpositions. Thus

$$\operatorname{sgn}(\sigma\mu) = (-1)^{n+m} = (-1)^n(-1)^m = \operatorname{sgn}(\sigma)\operatorname{sgn}(\mu)\,.$$

□

Chapter 3: Bilinear Forms and Quadratic Forms

3.1 Bilinear Forms

Bilinear forms play an important role in the definition of tensor algebras. Symmetric bilinear forms are also closely related to quadratic forms, which are an integral part of Clifford algebras. To prepare for these topics, we will cover bilinear forms and quadratic forms here.

3.1.1 Definition and Basic Properties

Definition 3.1. Given a real vector space V, a **bilinear form** is a function $B : V \times V \to \mathbb{R}$ that is linear in each coordinate. That is, for all $u, v, w \in V$ and $\lambda \in \mathbb{R}$,

$$B(u + v, w) = B(u, w) + B(v, w)\,,$$

$$B(u, v + w) = B(u, v) + B(u, w)\,, \text{ and}$$

$$B(\lambda u, v) = B(u, \lambda v) = \lambda B(u, v)\,.$$

Given a basis $\mathcal{B} = \{b_1, b_2, \ldots, b_n\}$ for V, a bilinear form B is completely determined once a value for $B(b_i, b_j)$ has been assigned for every pair of basis vectors. The bilinear form B can therefore be encoded as a matrix $M_{\mathcal{B}} = (a_{ij})$ by setting $a_{ij} = B(b_i, b_j)$. The action of B on two vectors $x, y \in V$ is then given by

$$B(x, y) = [x]_{\mathcal{B}}^t M_{\mathcal{B}} [y]_{\mathcal{B}}\,, \tag{3.1}$$

where $[x]_{\mathcal{B}}$ and $[y]_{\mathcal{B}}$ are the column vectors x and y in coordinate form relative to the basis

$\mathcal{B}$.

The matrix representation of B will depend on the basis chosen. However, the matrix $M_{\mathcal{B}}$ (representing B relative to the basis $\mathcal{B}$) and the matrix $M_{\mathcal{C}}$ (representing B relative to the basis $\mathcal{C}$) are related as *congruent* matrices.

Definition 3.2. Matrices A and B from $\mathbb{R}(n)$ are said to be **congruent** if there exists an invertible matrix P, also in $\mathbb{R}(n)$, such that

$$A = P^t B P \, .$$

The congruence of $M_{\mathcal{B}}$ and $M_{\mathcal{C}}$ is seen in the following way. Let $[b_i]_{\mathcal{C}}$ be the coordinate representation of the basis vector b_i using the basis $\mathcal{C}$. Let $M_{\mathcal{C},\mathcal{B}}$ be the coordinate transform matrix from basis $\mathcal{C}$ to basis $\mathcal{B}$. Then,

$$[b_i]_{\mathcal{B}} = M_{\mathcal{C},\mathcal{B}}[b_i]_{\mathcal{C}}$$

and

$$a_{ij} = [b_i]^t_{\mathcal{B}} M_{\mathcal{B}} [b_j]_{\mathcal{B}} = [b_i]^t_{\mathcal{C}} M^t_{\mathcal{C},\mathcal{B}} M_{\mathcal{B}} M_{\mathcal{C},\mathcal{B}} [b_j]_{\mathcal{C}} \, .$$

Thus, $M_{\mathcal{C}} = M^t_{\mathcal{C},\mathcal{B}} M_{\mathcal{B}} M_{\mathcal{C},\mathcal{B}}$.

There are two types of bilinear forms that will be of importance in the development of Clifford algebras. They are the *symmetric* and *anti-symmetric* bilinear forms, defined below.

Definition 3.3. Let V be a real vector space.

1. A bilinear form is called **symmetric** if, for all elements $x, y \in V$, $B(x, y) = B(y, x)$.

2. A bilinear form is called **anti-symmetric** or **skew-symmetric** if, for all elements $x, y \in V$, $B(x, y) = -B(y, x)$.

The defining characteristic of anti-symmetric bilinear forms, which is $B(x, y) = -B(y, x)$, is equivalent to the condition that $B(x, x) = 0$, as shown in the next proposition.

Proposition 3.1. Let B be a bilinear form on a real vector space V. Then $B(x,y) = -B(y,x)$ for all x and y if and only if $B(x,x) = 0$ for all x.

Proof. Assume $B(x,x) = 0$ for all $x \in V$. Then for any $x, y \in V$

$$0 = B(x+y, x+y) = B(x,x) + B(y,y) + B(x,y) + B(y,x)$$

$$= B(x,y) + B(y,x)$$

and so $B(x,y) = -B(y,x)$.

Conversely, assume $B(x,y) = -B(y,x)$ for all $x, y \in V$. Taking $y = x$ we obtain $B(x,x) = -B(x,x)$ from which we see that $B(x,x) = 0$. □

Given a real vector space V and a basis $\mathcal{B}$, it follows from Definition 3.3 and Equation 3.1 that if a bilinear form B is symmetric, then its corresponding matrix $M_{\mathcal{B}}$ will be symmetric. Likewise, if B is anti-symmetric, then $M_{\mathcal{B}}$ will be an anti-symmetric (skew-symmetric) matrix.

3.1.2 Inner Products

An inner product is a particular type of bilinear form. It is often denoted by angular brackets $\langle \cdot\, , \cdot \rangle$. Sometimes the notation used is that of a dot $\cdot$ between the vectors on which the inner product acts (e.g., $x \cdot y$). Another common name, *dot product*, is due to this last notation.

Definition 3.4 (inner product)**.** An inner product on a real vector space V is defined by the following properties. For all $v, w \in V$:

1. $\langle v\, , w \rangle = \langle w\, , v \rangle$,
2. $\langle v\, , v \rangle \geq 0$,
3. $\langle v\, , v \rangle = 0 \iff v = 0$.

The inner product is an example of a symmetric bilinear form. The term *positive definite* is used to describe properties 2 and 3.

Example 3.1. An inner product can be defined on the vector space $\mathbb{R}^n$ whereby for each $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ in $\mathbb{R}^n$

$$\langle x\,, y\rangle = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n\,.$$

This particular inner product is known as the *standard inner product* on $\mathbb{R}^n$. In the context of bilinear forms, we will reserve the angular brackets $\langle \cdot\,, \cdot\rangle$ exclusively for dealing with this special case.

Definition 3.5 (norm)**.** A map $\|\cdot\| : V \to \mathbb{R}$ is a norm if, for all $x, y \in V$ and any $r \in \mathbb{R}$, the following properties hold:

1. (*positive homogeneity*) $\|rx\| = |r|\,\|x\|$;
2. (*definite*) $\|x\| = 0 \iff x = 0$;
3. (*triangle inequality*) $\|x + y\| \leq \|x\| + \|y\|$.

Norm properties 2 and 3 imply that $\|x\| \geq 0$ for all x. Given an inner product on a real vector space V, it is always possible to define a norm for that space by

$$\|x\| = \sqrt{\langle x\,, x\rangle}\,, \quad \text{for all } x \in V\,.$$

A vector space V combined with an inner product defined on V is known as an *inner product space.*

Example 3.2. The vector space $\mathbb{R}^n$ combined with the standard inner product is an inner product space referred to as *Euclidean space.* The inner product induces the *standard* or *Euclidean norm,*

$$\|x\| = \sqrt{\langle x\,, x\rangle} = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}\,.$$

For $n = 3$, the standard inner product and norm give rise to all the familiar aspects of 3-dimensional Euclidean geometry. The familiar concept of vector length is conveyed by the norm and the angle between two vectors is defined using the inner product. The norm and inner product allow us to generalize the notions of length and angle to dimensions higher than three.

3.2 Quadratic Forms

Definition 3.6. Given a real vector space V, a **quadratic form** is a map $Q : V \to \mathbb{R}$ such that for all $x, y \in V$ and $r \in \mathbb{R}$,

1. $Q(rx) = r^2 Q(x)$, and
2. the map $B_Q(x, y) = \frac{1}{2}[Q(x + y) - Q(x) - Q(y)]$ is a symmetric bilinear form.

A quadratic form is said to be *nondegenerate* if $Q(x) = 0$ implies that $x = 0$. If there exists a nonzero x for which $Q(x) = 0$, then Q is called *degenerate*.

Example 3.3. The *standard norm* $\|\cdot\|$ on $\mathbb{R}^n$ is defined for $x = (x_1, x_2, \ldots, x_n)$ as

$$\|x\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}\,.$$

Note that $\|x\|^2 = \langle x\,, x\rangle$. From this we deduce that

$$\|x + y\|^2 = \langle x + y\,, x + y\rangle = \langle x\,, x\rangle + \langle y\,, y\rangle + 2\langle x\,, y\rangle\,,$$

and we obtain

$$\frac{1}{2}\left[\|x + y\|^2 - \|x\|^2 - \|y\|^2\right] = \langle x\,, y\rangle\,,$$

which, of course, is a symmetric bilinear form. Thus, $\|\cdot\|^2$ is a quadratic form and the standard inner product is its associated bilinear form.

Example 3.3 demonstrates a relationship between bilinear forms and quadratic forms that is true more generally. That is, given a symmetric bilinear form B, then $Q(x) = B(x, x)$ is a quadratic form. Furthermore, B and B_Q, the bilinear form associated with Q from property 2 of Definition 3.6, are one and the same, as the following shows:

$$\begin{aligned} B_Q(x,y) &= \frac{1}{2}[Q(x+y) - Q(x) - Q(y)] \\ &= \frac{1}{2}\big[B(x+y,\, x+y) - B(x,x) - B(y,y)\big] \\ &= \frac{1}{2}\big[B(x,y) + B(y,x)\big] = B(x,y)\,. \end{aligned}$$

In the context of quadratic forms, "form" refers to a homogeneous polynomial [Rom08]. Examining the matrix representation of a quadratic form acting on a vector x, we have

$$Q(x) = [x]_{\mathcal{B}}^t M_{\mathcal{B}} [x]_{\mathcal{B}} = \sum_{i,j} a_{ij} x_i x_j\,.$$

In this guise it is seen that a quadratic form is indeed a quadratic polynomial in the variables x_i and x_j. Clearly the polynomial will depend on the basis chosen. However, there is an important property of quadratic forms that remains invariant under changes of basis. This invariant will allow for categorizing quadratic forms no matter what basis is chosen for the underlying vector space. First, we introduce *orthogonal matrices*, which will be used in exposing the invariant.

Definition 3.7. An **orthogonal matrix** M is an $n \times n$ matrix for which the transpose is equal to the inverse, i.e., $M^t = M^{-1}$.

Theorem 3.2 (Spectral Theorem)**.** Let $M \in \mathbb{R}(n)$ be a real symmetric matrix. Then there exists a real orthogonal matrix P and a real diagonal matrix D such that $M = PDP^t$. The columns of P consist of eigenvectors of M and the diagonal of D consists of the eigenvalues of M.

Proof. The proof follows an analytical approach using Lagrange multipliers. Let $E_1 = \{x \in \mathbb{R}^n \mid \sum_{i=1}^n x_i^2 = 1\}$ and $f(x) = \langle Mx, x\rangle$. Set E_1 is the unit sphere in $\mathbb{R}^n$ and is compact. The function f is continuous and real-valued so f achieves a maximum on E_1. Let $v^{(1)}$ be the point at which f attains its maximum on E_1. Since f and the constraint function $\sum_{i=1}^n x_i^2 - 1$ are C^1 functions (in fact, they are C^∞), by the Lagrange multiplier theorem there exists a real number λ_1 such that

$$\nabla f(v^{(1)}) = \lambda_1 \nabla \left(\sum_{i=1}^n x_i^2 - 1 \right) \Bigg|_{x=v^{(1)}} = 2\lambda_1 v^{(1)} .$$

For fixed k note that

$$f(x) = \sum_{i,j=1}^n M_{ij} x_i x_j$$

$$= M_{kk} x_k^2 + \sum_{j \neq k} \left(M_{jj} x_j^2 + M_{jk} x_j x_k + M_{kj} x_k x_j \right) + \sum_{i,j \neq k} M_{ij} x_i x_j .$$

Therefore,

$$\frac{\partial f}{\partial x_k} = 2M_{kk} x_k + \sum_{j \neq k} M_{jk} x_j + \sum_{j \neq k} M_{kj} x_j = 2 \sum_{j=1}^n M_{kj} x_j$$

$$= 2(Mx)_k ,$$

where the term after the last equality represents the k^{th} component of the vector Mx. Since this holds for any fixed k, it follows that

$$\nabla f(x) = 2Mx$$

and therefore that

$$2Mv^{(1)} = 2\lambda_1 v^{(1)}\,.$$

Now consider $E_2 \subset E_1$ where $E_2 = \{x \in \mathbb{R}^n \mid \sum_{i=1}^n x_i^2 \text{ and } \langle x, v^{(1)}\rangle = 0\}$. E_2 is not empty and is compact, so again f achieves a maximum on E_2, say at the point $v^{(2)}$. A second constraint on f has been introduced for this domain. So, at $v^{(2)}$ there exists two Lagrange multipliers σ and λ_2 such that

$$\nabla f(v^{(2)}) = \lambda_2 \nabla \left(\sum_{i=1}^{n} x_i^2 - 1 \right) \Bigg|_{x=v^{(2)}} + \sigma \nabla \left(\langle x, v^{(1)} \rangle \right) \Big|_{x=v^{(2)}}\,.$$

Applying the same methods to this last Lagrangian as were used on the Lagrangian for E_1 we get

$$2Mv^{(2)} = 2\lambda_2 v^{(2)} + \sigma v^{(1)}\,. \tag{3.2}$$

Taking the inner product of $2Mv^{(2)}$ with $v^{(1)}$ gives

$$\langle 2Mv^{(2)}, v^{(1)}\rangle = \langle 2v^{(2)}, M^t v^{(1)}\rangle = \langle 2v^{(2)}, Mv^{(1)}\rangle$$

$$= 2\lambda_1 \langle v^{(2)}, v^{(1)}\rangle = 0\,.$$

Taking the inner product again, but this time substituting Equation 3.2 reveals

$$\langle 2Mv^{(2)}, v^{(1)}\rangle = \langle 2\lambda_2 v^{(2)} + \sigma v^{(1)}, v^{(1)}\rangle = \sigma = 0\,,$$

and therefore that

$$Mv^{(2)} = \lambda_2 v^{(2)}\,.$$

Proceeding by induction, the process produces the eigenvectors $v^{(i)}$ and their corresponding eigenvalues λ_i. Now, f attains its maximum on E_i at

$$f(v^{(i)}) = \langle Mv^{(i)}, v^{(i)}\rangle = \lambda_i\,,$$

that is,

$$\lambda_i = \max\{\langle Mx, x\rangle \mid x \in E_i\}\,.$$

Since $E_1 \supset E_2 \supset \cdots \supset E_n$ it follows that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$.

Let

$$P = (v^{(1)} v^{(2)} \cdots v^{(n)}) \quad \text{and} \quad D = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix}.$$

Then

$$MP = (Mv^{(1)} Mv^{(2)} \cdots Mv^{(n)}) = (\lambda_1 v^{(1)} \lambda_2 v^{(2)} \cdots \lambda_n v^{(n)}) = PD\,.$$

By construction, P is an orthogonal matrix, however, so $P^{-1} = P^t$, thus

$$M = PDP^t\,.$$

□

There is also an important congruence relation between any symmetric real matrix and a specific kind of diagonal matrix, which is detailed in the next theorem.

Theorem 3.3 (Sylvester's Law of Inertia)**.** *Let S be a real symmetric matrix. Then there exist unique numbers p, m, and z such that S is congruent to the diagonal matrix X given* by

$$X = \begin{pmatrix} 1 & & & & & & & & \\ & \ddots & & & & & & & \\ & & 1 & & & & & & \\ & & & -1 & & & & & \\ & & & & \ddots & & & & \\ & & & & & -1 & & & \\ & & & & & & 0 & & \\ & & & & & & & \ddots & \\ & & & & & & & & 0 \end{pmatrix} \tag{3.3}$$

where the number of diagonal entries with value 1 is p, the number of diagonal entries with value -1 is m and the number of diagonal entries with value 0 is z.

Proof. Take any diagonal real matrix D which contains d_i as the i^{th} diagonal entry. Construct the diagonal real matrix Q where the i^{th} diagonal entry is given by

$$q_i = \begin{cases} 1/\sqrt{|d_i|} & \text{if } d_i \neq 0\,, \\ 1 & \text{if } d_i = 0\,. \end{cases}$$

The matrix $H = (h_{ij}) = QDQ$ has only ones, negative ones, and/or zeros along the diagonal. Note since Q is symmetric, $Q^t = Q$, and hence H and D are in congruence.

Reordering the diagonal entries of H can be achieved through elementary row and column operations. Suppose that the diagonal entries h_{ii} and h_{jj} are to be transposed resulting in the matrix $H' = (h'_{ij})$. Then $h'_{ii} = h_{jj}$ and $h'_{jj} = h_{ii}$. It follows that the i^{th} and j^{th} rows of H' are equal to the j^{th} and i^{th} rows of H, respectively. The relation between the i^{th} and j^{th} columns of H' and H is similar. There exists an elementary matrix T that affects this change in both the rows and the columns when H is conjugated with T, that is,

$$H' = THT\,.$$

Therefore, a matrix X with the form of Equation 3.3 can be obtained from H by repeated conjugation with the appropriate elementary matrices. Because T only transposes two rows (columns), it is symmetric and hence, X is congruent to H.

It follows from the Spectral Theorem that any symmetric matrix S is congruent to a diagonal matrix D, which is in turn congruent to a diagonal matrix X that has the form given by Equation 3.3.

Suppose that a symmetric $n \times n$ real matrix S is congruent to two matrices X and Y that both have the form given in Equation 3.3, where X has p ones, m negative ones and z zeros and Y has p' ones, m' negative ones and z' zeros.

On the real vector space $\mathbb{R}^n$, let X represent the bilinear form B with respect to the basis

$$\mathcal{B} = \{u_1, \ldots, u_p, v_1, \ldots, v_m, w_1, \ldots, w_z\}\,.$$

Since Y is congruent to X, Y also represents B relative to another basis

$$\mathcal{C} = \{u'_1, \ldots, u'_{p'}, v'_1, \ldots, v'_{m'}, w'_1, \ldots, w'_{z'}\}\,.$$

Congruence between X and Y additionally implies they have the same rank, i.e., $p + m = p' + m'$ so that $z = z'$.

Now, for any nonzero vector $a \in \operatorname{span}(u_1, \ldots, u_p)$,

$$B(a,a) = B\left(\sum_{i=1}^{p} \alpha_i u_i, \sum_{j=1}^{p} \alpha_j u_j\right) = \sum_{i,j} \alpha_i \alpha_j \delta_{ij} = \sum_{i=1}^{p} \alpha_i^2 > 0\,.$$

If instead one chooses a vector $b \in \mathrm{span}(v'_1, \ldots, v'_{m'}, w'_1, \ldots, w'_{z'})$, then

$$\begin{aligned} B(b,b) &= B\left(\sum_{i=1}^{m'} \lambda_i v'_i + \sum_{j=1}^{z'} \mu_j v'_j, \sum_{k=1}^{m'} \lambda_k v'_k + \sum_{\ell=1}^{z'} \mu_\ell v'_\ell\right) \\ &= B\left(\sum_{i=1}^{m'} \lambda_i v'_i, \sum_{k=1}^{m'} \lambda_k v'_k\right) + B\left(\sum_{j=1}^{z'} \mu_j v'_j, \sum_{\ell=1}^{z'} \mu_\ell v'_\ell\right) = -\sum_{i,k} \lambda_i \lambda_k \delta_{ik} = -\sum_{i=1}^{m'} \lambda_i^2 \leq 0\,. \end{aligned}$$

Therefore, a and b must reside in disjoint subspaces, and so

$$p + m' + z' = p + (n - p') \leq n \quad \Rightarrow \quad p \leq p'\,.$$

A similar procedure yields $p' \leq p$, so $p = p'$. It follows that $m = m'$. □

Chapter 4: Algebras Defined by a Universal Property

In this chapter the concept of the universal property is introduced. The first section defines, generally, what a universal property is. The following sections employ different universal properties to define mathematical structures that will be critical in understanding the Clifford algebra; they are the tensor product, the tensor algebra and the exterior algebra. Later, in Chapter 5 we will define the Clifford algebra by means of a universal property. It should be pointed out that there are several other ways to define Clifford algebras. Interested readers should consult [Lou01].

4.1 The Universal Property

Definition 4.1. Let A be a set, $\mathcal{S}$ a collection of sets, and $\mathcal{F}$ a collection of functions that map from A to a set in $\mathcal{S}$. Let $\mathcal{H}$ be a collection of functions from a member of $\mathcal{S}$ to some set, also in $\mathcal{S}$. Assume $\mathcal{F}$ and $\mathcal{H}$ have the following characteristics:

1. $\mathcal{H}$ is closed under composition of functions, provided the composition is defined,
2. if Id is the identity function on some set $S \in \mathcal{S}$, then $\text{Id} \in \mathcal{H}$, and
3. for any $\tau \in H$ and $f \in \mathcal{F}$, if the composition $\tau \circ f$ is defined, then $\tau \circ f$ is an element of $\mathcal{F}$.

Consider any set $X \in \mathcal{S}$, and any $g \in \mathcal{F}$ that maps $A \to X$, and call them a generic set and a generic function, respectively. Likewise, call the pair (X, g) a *generic pair*. A set $U \in \mathcal{S}$ and a function $f \in \mathcal{F}$ will be called a universal set and a universal function, respectively, if for each generic pair (X, g) there exists a unique $\tau \in \mathcal{H}$ such that

$$g = \tau \circ f\,.$$

In this case, the pair (U, f) is called a *universal pair* for $(\mathcal{F}, \mathcal{H})$ and it is said to have the *universal property* for $\mathcal{F}$ as *measured by* $\mathcal{H}$.

The relations between the various functions and sets can be summarized with the commuting diagram in Figure 4.1. Four important constructions serve as examples of the universal property. These are the tensor product, the tensor algebra, the exterior algebra, and the Clifford algebra. The first three are considered in the remainder of this chapter; they will be crucial in developing the Clifford algebra, which is presented in the next chapter.

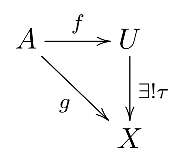

Figure 4.1: Commuting diagram demonstrating a general universal property.

4.2 The Tensor Product

4.2.1 Definition

Let $V_1, \ldots, V_n$ be real vector spaces and define the following sets:

$$A = V_1 \times \cdots \times V_n\,,$$

$$\mathcal{S} = \{W \mid W \text{ a real vector space}\}\,,$$

$$\mathcal{F} = \{f : V_1 \times \cdots \times V_n \to W \mid W \in \mathcal{S}\,,\ \text{and } f \text{ multilinear}\}\,,\ \text{and}$$

$$\mathcal{H} = \{\tau \mid \tau \text{ is a linear map between real vector spaces}\}\,.$$

It follows from linearity and multilinearity that the sets $\mathcal{H}$ and $\mathcal{F}$, respectively, meet the requirements laid out in the definition of the universal property.

Let (W, f) be a universal pair for $(\mathcal{F}, \mathcal{H})$. The set W is called a *tensor product* of $V_1, V_2, \ldots, V_n$. It is denoted $V_1 \otimes \cdots \otimes V_n$, or alternatively, $\bigotimes_{i=1}^{n} V_i$. The term "tensor product" gets double usage because, for any vectors $v_i \in V_i$, one can also form the tensor product of the vectors: $v_1 \otimes \cdots \otimes v_n$, and this object is an element of $V_1 \otimes \cdots \otimes V_n$. This element is defined by the universal function f:

$$v_1 \otimes \cdots \otimes v_n = f(v_1, \ldots, v_n)\,. \tag{4.1}$$

In general, an element of $V_1 \otimes \cdots \otimes V_n$ will be a linear combination of objects having the form of that on the left-hand side of Equation 4.1.

It follows from the multilinearity of f that

$$v_1 \otimes \cdots \otimes v_{i-1} \otimes (av_i + a'v_i') \otimes v_{i+1} \otimes \cdots \otimes v_n =$$

$$a\big(v_1 \otimes \cdots \otimes v_{i-1} \otimes v_i \otimes v_{i+1} \otimes \cdots \otimes v_n\big) + a'\big(v_1 \otimes \cdots \otimes v_{i-1} \otimes v_i' \otimes v_{i+1} \otimes \cdots \otimes v_n\big)\,.$$

The following notation will be used for the special case of the tensor product of a real vector space V with itself p times

$$T^p(V) = \underbrace{V \otimes \cdots \otimes V}_{p \text{ times}}, \text{ where } p \text{ is a non-negative integer.}$$

The case $p = 0$ gives the base field: $T^0(V) = \mathbb{R}$.

Although the name "tensor product" is used for both a product of vector spaces and for a product of vectors, it will typically be clear from context which type is intended. Sometimes "tensor product space" is used to refer to a tensor product of vector spaces.

4.2.2 Existence and Basis for the Tensor Product

The tensor product has been defined but it has yet to be shown to actually exist and this is where we next focus our attention. Let $A, B, \ldots, Z$ be a family of finite dimensional vector spaces. The symbols $A, B, \ldots, Z$ are chosen for notational convenience and not meant to imply there is one vector space for each letter of the alphabet. Take n to be the number of these vector spaces and $d_1, d_2, \ldots, d_n$ to be the dimensions of the vector spaces. Let $\{a_i\}_{1\leq i\leq d_1},\ \{b_j\}_{1\leq j\leq d_2},\ \ldots,\ \{z_k\}_{1\leq k\leq d_n}$ be the bases of $A, B, \ldots, Z$, respectively. Define f to be the function that maps each n-tuple $(a_i, b_j, \ldots, z_k) \in A \times B \times \cdots \times Z$ to the object represented by the symbol $a_i \otimes b_j \otimes \cdots \otimes z_k$. By Theorem 1.4, the collection $\mathcal{B} = \{a_i \otimes b_j \otimes \cdots \otimes z_k \mid 1 \leq i \leq d_1,\ 1 \leq j \leq d_2,\ \ldots,\ 1 \leq k \leq d_n\}$ forms a basis for a vector space W. The map f can be extended uniquely to a multilinear map from $A \times B \times \cdots \times Z$ to W.

Now, for any generic pair (X, g), define τ on $\mathcal{B}$ by

$$\tau(a_i \otimes \cdots \otimes z_k) = g(a_i, \ldots, z_k)\,. \tag{4.2}$$

By Theorem 1.5, extending τ by linearity to all of W results in a unique linear function $\tau : W \to X$ such that $\tau \circ f = g$.

We have shown that (W, f) is a universal pair and W is the tensor product. Furthermore, by definition, a basis for W is $\mathcal{B} = \{a_i \otimes b_j \otimes \cdots \otimes z_k \mid 1 \leq i \leq d_1,\ 1 \leq j \leq d_2,\ \ldots,\ 1 \leq k \leq d_n\}$. It follows from this that $\dim \bigotimes_{i=1}^{n} V_i = \prod_{i=1}^{n} \dim V_i$.

4.2.3 The Tensor Product Space as an Algebra

Strictly speaking, a tensor product is a vector space formed from other vector spaces, but often it is useful to impose more structure. This can be done readily if the tensor product's factors $A_1, A_2, \ldots, A_n$ are algebras in addition to being vector spaces. Then the tensor product space $A_1 \otimes A_2 \otimes \cdots \otimes A_n$ becomes an algebra by defining a multiplication operation that makes use of the multiplication rules of the constituent algebras $A_1, \ldots, A_n$. The multiplication rule for the tensor product space is demonstrated for $n = 2$. The rule for integer $n > 2$ follows by induction.

Let $\{e_i^1\}_i$ be a basis for A_1 and $\{e_j^2\}_j$ a basis for A_2. Multiplication is defined on general linear combinations of the e_i^1 and e_j^2 by

$$\Big(\sum_{i,j} a_{ij}\, e_i^1 \otimes e_j^2\Big)\Big(\sum_{k,\ell} b_{k\ell}\, e_k^1 \otimes e_\ell^2\Big) = \sum_{i,j,k,\ell} a_{ij} b_{k\ell} \big(e_i^1 e_k^1 \otimes e_j^2 e_\ell^2\big)\,.$$

The associativity, closure and scalar multiplication requirements of an algebra are automatically fulfilled due to $A_1, \ldots, A_n$ being algebras, and distribution over addition is implicit in the definition. Thus, the necessary requirements of an algebra are met. Of course, other multiplication rules can be defined, but this particular multiplication will be considered standard and will be implied unless a different rule is given.

Two tensor product spaces that contain the same factors, but which differ in the order in which the factors appear, are isomorphic. The next proposition, adapted from [Nor84], states this more rigorously.

Proposition 4.1. Let $\mathcal{T}_1 = A_1 \otimes \cdots \otimes A_n$ be a tensor product space. Let σ be a permutation on $\{1, \ldots, n\}$ and let $\mathcal{T}_2 = A_{\sigma(1)} \otimes \cdots \otimes A_{\sigma(n)}$ be the tensor product space obtained by permuting the order of the factors in $\mathcal{T}_1$. Then $\mathcal{T}_1$ and $\mathcal{T}_2$ are isomorphic.

Proof. Let $g_1 : A_1 \times \cdots \times A_n \to A_{\sigma(1)} \otimes \cdots \otimes A_{\sigma(n)}$ be the multilinear map defined by $g_1(a_1, \ldots, a_n) = a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(n)}$, for all $a_i \in A_i$, where $1 \leq i \leq n$. By the universal property of the tensor product, there exists a unique linear τ_1, together with the universal function f_1, such that $\tau_1 \circ f_1 = g_1$. Let $g_2 : A_{\sigma(1)} \times \cdots \times A_{\sigma(n)} \to A_1 \otimes \cdots \otimes A_n$ be the multilinear map defined by $g_2(a_{\sigma(1)}, \ldots, a_{\sigma(n)}) = a_1 \otimes \cdots \otimes a_n$. In this case, the universal property yields a universal function f_2 and a unique linear τ_2 such that $\tau_2 \circ f_2 = g_2$. Figure 4.2 contains a commuting diagram showing the usage of the universal property and the relationships it produces. Now, for any $a_i \in A_i$,

$$f_1(a_1, \ldots, a_n) = a_1 \otimes \cdots \otimes a_n = \big(\tau_2 \circ f_2\big)(a_{\sigma(1)}, \ldots, a_{\sigma(n)}) \,, \quad \text{and}$$

$$f_2(a_{\sigma(1)}, \ldots, a_{\sigma(n)}) = a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(n)} = \big(\tau_1 \circ f_1\big)(a_1, \ldots, a_n) \,.$$

Together, these equalities show $\tau_2 \circ \tau_1$ and $\tau_1 \circ \tau_2$ to be the identities on $\mathcal{T}_1$ and $\mathcal{T}_2$, respectively. Thus, τ_1 is an isomorphism. □

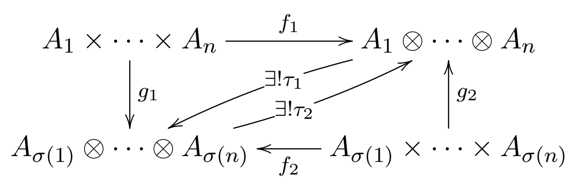

Figure 4.2: Proposition 4.1 states that permuting the order of the factors in a tensor product space results in a new tensor product space that is isomorphic to the original. The proof makes use of the tensor product's universal property twice. The commuting diagram illustrates this usage and the resulting relationships. Generic functions g_1 and g_2 are defined in such a way that the linear maps that factor them, τ_1 and τ_2, respectively, produce identity maps when composed as $\tau_1 \circ \tau_2$ and $\tau_2 \circ \tau_1$.

4.3 The Tensor Algebra

The tensor algebra will be defined by a universal property but before doing so, the concept of an *inclusion map* is introduced. An inclusion map will be part of the concrete universal pair used to demonstrate that the tensor algebra does in fact exist.

4.3.1 Inclusion Maps

An inclusion map $i : X \to Y$ is an injection from a set X into set Y. If X has additional structure defined on it, then i typically also preserves this structure in its mapping to $i(X)$. Intuitively, if a set X can be regarded as a subset of another set Y, then i is the map that sends each $x \in X$ to $x \in Y$. Strictly speaking, the element x and $i(x)$ may not be the same, but since i is a one-to-one correspondence between X and $i(X)$, element $i(x)$ amounts to a re-labeling of element x. In essence, then, X is contained in Y; the inclusion map isolates that subset of Y that is equivalent to X. The set X is said to be *embedded* in Y and i is sometimes called an *embedding*, particularly in the case that X has a structure defined on it and i preserves that structure.

The inclusion maps we will use will be between vector spaces and will preserve vector space structure. With this in mind, the definition of inclusion map given here is in terms of vector spaces.

Definition 4.2 (Inclusion map (between vector spaces))**.** Let X and Y be vector spaces. An **inclusion map** is a linear injection $i : X \to Y$.

An example illustrates an inclusion map.

Example 4.1 (Real line embedded in $\mathbb{R}^n$)**.** The map $i : \mathbb{R} \to \mathbb{R}^n$ that sends $r \mapsto (r, 0, 0, \ldots, 0) \in \mathbb{R}^n$ is an inclusion map. The Cartesian product

$$\mathbb{R} \times \underbrace{\{0\} \times \cdots \times \{0\}}_{n-1 \text{ times}}$$

is the subset of $\mathbb{R}^n$ that is being viewed as a "copy" of $\mathbb{R}$ embedded in $\mathbb{R}^n$.

Because we think of $i(X)$ as being a "copy" of X embedded in Y, we often denote the element $i(x)$ as simply x. This serves to reinforce the identification of $i(x)$ with x and also unencumbers notation in equations, however, it can also be a cause for confusion since $i(x)$

and x are elements of different sets. A note will be made in the accompanying text when we choose to adopt this notation.

4.3.2 The Universal Property of the Tensor Algebra

Let V be a real vector space, and define the following sets:

$$A = V,$$

$$\mathcal{S} = \{W \mid W \text{ a real unital algebra}\},$$

$$\mathcal{F}_{T(V)} = \{f : V \to W \mid W \in \mathcal{S}\,,\ \text{and } f \text{ linear}\}\,,\ \text{and}$$

$$\mathcal{H} = \{\tau \mid \tau \text{ is an algebra homomorphism}\}\,.$$

The sets $\mathcal{F}$ and $\mathcal{H}$ meet the conditions given in the universal property definition due to their linear and homomorphism properties, respectively. Take $T(V)$ to be a real unital algebra in $\mathcal{S}$ and $i : V \to T(V)$ a linear function in $\mathcal{F}$. If the pair $\big(T(V), i\big)$ is a universal pair, then $T(V)$ is called the *tensor algebra* of V and so for any generic pair (W, g) there exists a unique algebra homomorphism $\phi \in \mathcal{H}$ such that $g = \phi \circ i$.

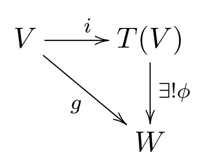

Figure 4.3: A commuting diagram illustrating the universal property of the tensor algebra $T(V)$. The tensor algebra is defined in Equation 4.3 and the universal function i is the inclusion map which embeds V in $T(V)$.

4.3.3 Existence of the Tensor Algebra

For any real vector space V with basis $\{e_1, \dots, e_n\}$, define $T(V)$ to be the direct sum

$$T(V) = \bigoplus_{p=0}^{\infty} T^p(V) = \mathbb{R} \;\oplus\; V \;\oplus\; (V \otimes V) \;\oplus\; \cdots \,, \tag{4.3}$$

and let i be the inclusion map that embeds V in $T(V)$. We next show that $(T(V), i)$ is a universal pair with respect to the above universal property.

The product on $T(V)$ is the tensor product: for any $x \in T^p(V)$ and any $y \in T^q(V)$, the product $xy = x \otimes y \in T^{p+q}(V)$. When $p = 0$, x is an element of the embedding of $\mathbb{R}$, and is a scalar multiple of the algebra's unit. With $z \in T^r(V)$, the required distributive property dictates that the product $x(y+z) = x \otimes y + x \otimes z$ and $(x+y)z = x \otimes z + y \otimes z$.

For any $p \geq 1$, identify each $v_1 \otimes \cdots \otimes v_p \in T^p(V)$ with its image in the embedded $T^p(V) \subset T(V)$. Then in particular, given the basis of $T^p(V)$ described in Section 4.2.2, each $e_{i_1} \otimes \cdots \otimes e_{i_p}$ is the embedded image of a basis vector of $T^p(V)$. Then for any generic pair (W, g) it is possible to define the map $\phi : T(V) \to W$ by

$$\phi(e_{i_1} \otimes \cdots \otimes e_{i_p}) = g(e_{i_1}) \cdots g(e_{i_p})$$

and extending linearly. Theorem 1.5 ensures ϕ is unique. With this definition, for any $v \in V$, $\phi(i(v)) = g(v)$ and furthermore

$$\phi(v_1 \otimes \cdots \otimes v_p) = g(v_1) \cdots g(v_p) = \phi(v_1) \cdots \phi(v_p) \,,$$

thus ϕ is a homomorphism. Therefore, $\big(T(V), i\big)$ is indeed a universal pair and $T(V)$ is the tensor algebra.

4.3.4 A Basis for the Tensor Algebra

Let V be a finite dimensional vector space and $\{e_1, e_2, \ldots, e_n\}$ be a basis for V. Referring to Equation (4.3) and Definition 1.8 for the direct sum, it is seen that the tensor algebra consists of a set of functions

$$T(V) = \left\{ t : I \to \bigcup_{i \in I} T^i(V) \;\middle|\; t(i) \in T^i(V) \text{ and } t \text{ has finite support} \right\} \tag{4.4}$$

where $I = \mathbb{N} \cup \{0\}$.

Now consider the subset of $T(V)$ given by

$$T_p = \left\{ t : I \to \bigcup_{i \in I} T^i(V) \;\middle|\; t(i) \in T^i(V) \text{ for } i = p \text{ and } t(i) = 0 \text{ for } i \neq p \right\}. \tag{4.5}$$

Note that the properties of scalar multiplication and addition of functions imply that T_p is a vector subspace of $T(V)$, and that $T_p \cap T_q = \{0\}$ if $p \neq q$. Furthermore, note that all of $T(V)$ can be obtained by taking finite linear combinations of functions from $\bigcup_{p \in I} T_p$. This also means that if B_p is a basis for the subspace T_p, then $\mathscr{B} = \bigcup_{p \in I} B_p$ spans $T(V)$ as well. The properties of the T_p ensure that the elements of $\mathscr{B}$ are linearly independent, thus it is a basis of $T(V)$.

There is one function in T_p for each element $v \in T^p(V)$, i.e., the function t_v that maps $p \mapsto v$. Thus, there is a 1-1 correspondence between T_p and $T^p(V)$. Therefore, the functions in T_p that serve as a basis for T_p are those that map p to the basis elements of $T^p(V)$. Using the 1-1 correspondence to identify $t_v \in T_p$ with $v \in T^p(V)$, the set $B_p = \{e_{j_1} \otimes e_{j_2} \otimes \cdots \otimes e_{j_p} \mid 1 \leq j_k \leq n\}$ is a basis for T_p. For $p = 0$, the basis set B_0 consists of the single element $1 \in T^0(V) = \mathbb{R}$. The basis $\mathscr{B}$ of $T(V)$, then, is given by

$$\mathscr{B} = \left\{ e_{j_1} \otimes e_{j_2} \otimes \cdots \otimes e_{j_p} \mid p \in \mathbb{N} \cup \{0\} \text{ and } 1 \leq j_k \leq n \right\}.$$

4.3.5 Comments

It is inferred from the preceding discussion in Section 4.3.4 that $T(V)$ can be identified with linear combinations of elements from $\bigcup T^p(V)$. If τ is a function in $T(V)$ that has finite support on the indices in the ordered n-tuple $(i_1, i_2, \ldots, i_n)$ such that $\tau(i_j) = v_{i_j} \in T^{i_j}(V)$, then τ will be denoted by $v_{i_1} + v_{i_2} + \cdots + v_{i_n}$. From here on, the elements of $T(V)$ will be written using this notation.

The tensor algebra is an example of a *graded algebra*, discussed in Definition 1.15. The tensor algebra will play a central role in constructing our next example of an algebra defined by a universal property: the exterior algebra.

4.4 The Exterior Algebra

4.4.1 The Universal Property of the Exterior Algebra

Let V be a real, finite-dimensional vector space, and define the following sets:

$$A = V,$$

$$\mathcal{S} = \{W \mid W \text{ a real unital algebra}\},$$

$$\mathcal{F}_\wedge = \{g : V \to W \mid W \in \mathcal{S}, g \text{ linear and } g(v)g(v) = 0 \;\forall\, v \in V\}, \text{ and}$$

$$\mathcal{H} = \{\tau \mid \tau \text{ is an algebra homomorphism}\}.$$

Let $f \in \mathcal{F}$ be a linear map from V to a real unital algebra $\bigwedge(V)$. If the pair $\big(\bigwedge(V), f\big)$ satisfies the universal property, then $\bigwedge(V)$ is called the *exterior algebra* of V and so for any generic pair (W, g) there exists a unique algebra homomorphism $\phi \in \mathcal{H}$ such that $\phi \circ f = g$. The relationships are summarized in the commuting diagram in Figure 4.4. To show the

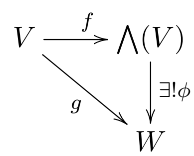

Figure 4.4: The universal property of the exterior algebra.

existence of $\bigwedge(V)$ we will form a quotient space from the tensor algebra $T(V)$ by an ideal J. We will define J in two equivalent ways. One definition facilitates the demonstration of the existence of $\bigwedge(V)$, and the other is used to show that $\bigwedge(V)$ is not a trivial construction. The first order of business is to introduce the two definitions of J and show their equality.

4.4.2 Definition and Equality of the Ideals

Let V be a real vector space of dimension n with basis $\{e_1, \ldots, e_n\}$, and let $T(V)$ be the tensor algebra of V. Recall from Section 4.3.4 that the set $\mathscr{B} = \big\{e_{i_1} \otimes \cdots \otimes e_{i_p} \;\big|\; p \in \{0, 1, 2, \ldots\} \text{ and } i_j \in \{1, 2, \ldots, n\} \text{ for } 1 \le j \le p\big\}$ is a basis for $T(V)$. A special note is made that in the basis elements, the factors e_{i_j} have no particular order.

For $q > 0$, let $\mathscr{B}_q = \big\{e_{i_1} \otimes \cdots \otimes e_{i_q} \mid i_j \in \{1, 2, \ldots, n\} \text{ for } 1 \le j \le q\big\}$ be the subset of $\mathscr{B}$ consisting of those basis elements that have exactly q factors. Let S_q be the set of permutations on $\{1, 2, \ldots, q\}$. Given any permutation $\sigma_q \in S_q$, we define the function $\lambda_{\sigma_q} : \mathscr{B}_q \to \mathscr{B}_q$ such that $\lambda_{\sigma_q}(e_{i_1} \otimes \cdots \otimes e_{i_q}) = e_{i_{\sigma_q(1)}} \otimes \cdots \otimes e_{i_{\sigma_q(q)}}$. Thus, λ_{σ_q} permutes the factors based on their order in the tensor product.

Define the ideals J_1 and J_2 by $J_1 = \langle v \otimes v \mid v \in V \rangle$ and $J_2 = \langle \operatorname{sgn}(\sigma_q)\lambda_{\sigma_q}(e) - e \mid e \in$

$\mathscr{B}_q$, $\sigma_q \in S_q\,, q \geq 1\rangle$.

Theorem 4.2. The ideals J_1 and J_2 are equal.

Proof. It is first demonstrated that $J_2 \subseteq J_1$. Consider the vector $e_j + e_k \in V$. We have

$$J_1 \ni (e_j + e_k) \otimes (e_j + e_k) - e_j \otimes e_j - e_k \otimes e_k = e_k \otimes e_j + e_j \otimes e_k\,.$$

The ideal J_1 is closed with respect to left and right multiplication by elements of $T(V)$, so any element of the form

$$e_{\ell_1} \otimes \cdots \otimes e_{\ell_q} \otimes (e_k \otimes e_j + e_j \otimes e_k) \otimes e_{m_1} \otimes \cdots \otimes e_{m_r} =$$

$$e_{\ell_1} \otimes \cdots \otimes e_{\ell_q} \otimes e_k \otimes e_j \otimes e_{m_1} \otimes \cdots \otimes e_{m_r} + e_{\ell_1} \otimes \cdots \otimes e_{\ell_q} \otimes e_j \otimes e_k \otimes e_{m_1} \otimes \cdots \otimes e_{m_r}$$

is an element of J_1, where values of 0 for q or r indicate no factors multiplied on the left or right, respectively. The right-hand side of the above expression consists of a sum of two terms, both of which have the general form of basis elements of $T(V)$ that have at least two factors. Furthermore, both terms are the same except for a transposition of the neighboring factors e_j and e_k. We thus conclude that for any basis element $e^{(2)}$ of $T(V)$ with at least two factors,

$$\lambda_\tau e^{(2)} + e^{(2)} \in J_1$$

where τ is a transposition that interchanges two neighboring factors of $e^{(2)}$.

By Corollary 2.3 and Lemma 2.4, we can write any permutation as a product of transpositions τ_j that interchange neighboring elements only, i.e, a permutation $\sigma = \tau_k \cdots \tau_1$. Thus, for the examples $k = 2$ and $k = 3$,

$$\lambda_{\tau_2\tau_1} e^{(2)} - e^{(2)} = (\lambda_{\tau_2\tau_1} e^{(2)} + \lambda_{\tau_1} e^{(2)}) - (\lambda_{\tau_1} e^{(2)} + e^{(2)}) \in J_1 \quad \text{and}$$

$$\lambda_{\tau_3\tau_2\tau_1} e^{(2)} + e^{(2)} = (\lambda_{\tau_3\tau_2\tau_1} e^{(2)} + \lambda_{\tau_2\tau_1} e^{(2)}) - (\lambda_{\tau_2\tau_1} e^{(2)} + \lambda_{\tau_1} e^{(2)}) + (\lambda_{\tau_1} e^{(2)} + e^{(2)}) \in J_1\,,$$

since each expression in parentheses on the right-hand sides of the equations is in the ideal J_1. In general, then,

$$\lambda_{\tau_k \tau_{k-1} \cdots \tau_1} e^{(2)} - (-1)^k e^{(2)} = \lambda_\sigma e^{(2)} - (-1)^k e^{(2)} \in J_1 .$$

But $(-1)^k = \text{sgn}(\sigma)$ so

$$(-1)^k \big(\lambda_\sigma e^{(2)} - (-1)^k e^{(2)} \big) = \text{sgn}(\sigma) \lambda_\sigma e^{(2)} - e^{(2)} \in J_1 .$$

Now, for basis elements $e^{(1)}$ in $\mathscr{B}$ consisting of only one factor, the only permutation possible is the identity permutation σ_{id} and

$$\text{sgn}(\sigma_{\text{id}}) \lambda_{\sigma_{\text{id}}} e^{(1)} - e^{(1)} = e^{(1)} - e^{(1)} = 0 \in J_1 .$$

Therefore, the entire generating set of J_2 is contained in J_1. As noted in Section 1.3, the intersection of all ideals containing this generating set yields J_2, therefore, $J_2 \subseteq J_1$.

To show that $J_1 \subseteq J_2$, first note that any arbitrary $v \in V$ can be written as the linear combination $v = \sum_{i=1}^n a_i e_i$ and that

$$v \otimes v = \sum_{j,k=1}^{n} a_j a_k (e_j \otimes e_k) = \sum_{j=1}^{n} a_j^2 (e_j \otimes e_j) + \sum_{\substack{j,k=1 \\ j \neq k}}^{n} a_j a_k (e_j \otimes e_k + e_k \otimes e_j) . \tag{4.6}$$

The two-factor product $e_j \otimes e_j$ has the form $e_{i_1} \otimes e_{i_2}$ and so for $\sigma = (1 \;\; 2)$,

$$J_2 \ni \text{sgn}(\sigma) \lambda_\sigma (e_j \otimes e_j) - (e_j \otimes e_j) = -2(e_j \otimes e_j) .$$

Thus, $e_j \otimes e_j \in J_2$. Similarly, the two-factor product $e_j \otimes e_k$ also has the form $e_{i_1} \otimes e_{i_2}$ and so

$$J_2 \ni \text{sgn}(\sigma) \lambda_\sigma (e_j \otimes e_k) - e_j \otimes e_k = -(e_k \otimes e_j \; + \; e_j \otimes e_k) .$$

Each term on the right-hand side of Equation 4.6 is therefore a member of J_2 and so $v \otimes v \in J_2$ as well, hence $J_1 \subseteq J_2$. □

4.4.3 Existence and Nontriviality of the Exterior Algebra

First, the existence of the exterior algebra is shown. Choose any generic pair $(W, g) \in \mathcal{S} \times \mathcal{F}_{\wedge}$, as in Section 4.4.1. Observe that g is also an element of $\mathcal{F}_{T(V)}$. Make use of the universal property of the tensor algebra $T(V)$ to obtain the unique algebra homomorphism ϕ such that $g = \phi \circ i$. Next, use the ideal $J = J_1 = J_2$ to form the quotient algebra $T(V)/J$, and let π be the canonical projection mapping.

For any $v \in V$,

$$v \otimes v = i(v)i(v) \;\Rightarrow\; \phi(v \otimes v) = \phi\big(i(v)\big)\phi\big(i(v)\big) = g(v)g(v) = 0\,.$$

Thus, the generating set for J_1 is a subset of $\ker \phi$. Since $\ker \phi$ is itself an ideal, $J_1 \subseteq \ker \phi$. By Theorem 1.7, there exists a unique algebra homomorphism ϕ_1 such that $\phi = \phi_1 \circ \pi$. Therefore, $g = \phi_1 \circ \pi \circ i$ and $\big(T(V)/J, \pi \circ i\big)$ is shown to be a universal pair. Consequently, $\bigwedge(V) = T(V)/J$. The various mappings used to demonstrate existence are illustrated in the commuting diagram of Figure 4.5.

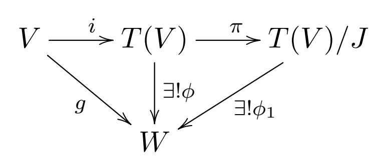

Figure 4.5: The commuting diagram illustrates the use of the tensor algebra's universal property in determining that $\big(T(V)/J,\, \pi \circ i\big)$ is a universal pair for the universal property of the exterior algebra. The unique algebra homomorphism ϕ_1 is obtained by use of Theorem 1.7.

That $\bigwedge(V) = T(V)/J$ is nontrivial, i.e. that J is not equal to $T(V)$, is shown using the generating set of J_2. Take any element of the generating set and multiply it on the left or right by a basis element of $T(V)$. The product is either another element in the generating set, provided the basis element contains no factor in common with the expression from the generating set, or the product is a finite sum of terms each with a repeated factor. In the latter case, the product is in J_1. Since J_1 and J_2 are equal, J_2 must be the vector space generated by the generating set of J_2 along with the elements of $T(V)$ that have at least one repeated factor. Thus, each basis element of J_2 has at least two factors from V and due to linear independence of the tensor algebra's basis elements, no nontrivial linear combination of elements of V is in the ideal. Therefore, $\pi \circ i$ is injective from V to $T(V)/J$.

4.4.4 A Basis for the Exterior Algebra

Let $\{e_1, \ldots, e_n\}$ be a basis for the real vector space V, and let $\mathscr{B}$ be the basis of $T(V)$ that was demonstrated in Section 4.3.4. Take $\mathscr{B}' = \left\{e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_p} \mid p \in \mathbb{N} \cup \{0\} \text{ and } 1 \leq i_1 < i_2 < \cdots < i_p \leq n\right\}$ be the subset of $\mathscr{B}$ whose elements are composed of factors with indices in strictly increasing order. No (nontrivial) linear combination of elements of $\mathscr{B}'$ is in the ideal J and so $\{\pi(b) \mid b \in \mathscr{B}'\}$ is a linearly independent subset of $\bigwedge(V)$.

Suppose e is an element of $\mathscr{B} \setminus \mathscr{B}'$ that has the repeated factors $e_{i_j} = e_{i_k}$. Let $\sigma = (j\ k)$ so that λ_σ interchanges e_{i_j} and e_{i_k}. Then $\lambda_\sigma e = e$ and $\operatorname{sgn}(\sigma) = -1$, so $-\lambda_\sigma e - e \in J$. Thus,

$$\pi(-\lambda_\sigma e - e) = 0 \quad \Rightarrow \quad -\pi(\lambda_\sigma e) = \pi(e) \quad \Rightarrow \quad -\pi(e) = \pi(e) \quad \Rightarrow \quad \pi(e) = 0\,.$$

Now suppose instead that e is an element of $\mathscr{B} \setminus \mathscr{B}'$ that has no repeated factors but whose factors are not ordered with increasing indices. Let σ be such that λ_σ acts to order the factors of e so that their indices are in strictly increasing order, i.e., $\lambda_\sigma e \in \mathscr{B}'$. Then

$\operatorname{sgn}(\sigma)\lambda_\sigma e - e \in J$, implying that

$$\operatorname{sgn}(\sigma)\pi(\lambda_\sigma e) - \pi(e) = 0 \quad \Rightarrow \quad \pi(e) = \operatorname{sgn}(\sigma)\pi(\lambda_\sigma e)\,.$$

We have thus shown that the image under π of any element of $T(V)$ can be written as a linear combination of the image under π of elements of $\mathscr{B}'$. Therefore, $\pi(\mathscr{B}')$ spans $T(V)/J$. Thus, the set $\pi(\mathscr{B}')$ is a basis for $\bigwedge(V)$, where $\mathscr{B}'$ is the subset of basis elements of $T(V)$ that only have factors with indices in strictly increasing order.

4.4.5 The Product of the Exterior Algebra

The set $T(V)/J$ automatically inherits a product structure from the canonical projection map π. The product, denoted by the symbol $\wedge$, is called the *exterior product*, or more colloquially, the *wedge product*. Every element of $T(V)/J$ is a linear combination of elements of the form $\pi(e_{i_1} \otimes \cdots \otimes e_{i_p})$, where $1 \leq i_1 < i_2 < \cdots < i_p \leq \dim V$. Because π is a homomorphism it follows that

$$\pi(e_{i_1} \otimes \cdots \otimes e_{i_p}) = \pi(e_{i_1}) \wedge \cdots \wedge \pi(e_{i_p}) = e_{i_1} \wedge \cdots \wedge e_{i_p}\,,$$

where in the last equality use has been made of an inclusion map to identify each $v \in V$ with $\pi(v) \in T(V)/J$.

Note that the basis elements $e_{i_1} \otimes \cdots \otimes e_{i_p}$ of $T(V)$ do not carry the restriction that their factors' subscripts be in increasing order, unlike the basis elements of $\bigwedge(V)$. However, it still holds that $\pi(e_{i_1} \otimes \cdots \otimes e_{i_p}) = e_{i_1} \wedge \cdots \wedge e_{i_p}$. From this and the properties of ideal J, it follows that

1. if $i_k = i_\ell$ for some $k \neq \ell$, then $e_{i_1} \wedge \cdots \wedge e_{i_p} = 0$, and

2. if σ permutes $\{1, \ldots, p\}$ $(1 \leq p \leq \dim V)$, then

$$e_{i_{\sigma(1)}} \wedge \cdots \wedge e_{i_{\sigma(p)}} = \operatorname{sgn}(\sigma)\, e_{i_1} \wedge \cdots \wedge e_{i_p} .$$

   In particular, we take σ to permute $\{1, \ldots, p\}$ such that $1 \leq i_{\sigma(1)} < i_{\sigma(2)} < \cdots < i_{\sigma(p)} \leq \dim V$.

These properties allow us to map any basis element of $T(V)$, which may contain unordered or repeated factors, to the appropriate element in $\bigwedge(V)$, which must not have unordered or repeated factors in the basis elements. Likewise, these properties allow us to determine the product of two basis elements of $\bigwedge(V)$ when the elements have a factor in common, or when their multiplication results in unordered factors.

Part II

Clifford Algebras and Their Classification

Chapter 5: The Clifford Algebra

In this chapter we come finally to introducing the Clifford algebra. We will give its definition in terms of a universal property from which it will follow that a Clifford algebra exists given a real, finite-dimenional vector space V and a nondegenerate quadratic form, and that it has dimension 2^n, where $n = \dim V$. It will be shown that a Clifford alebra is a $\mathbb{Z}_2$-graded algebra and this fact will be used to show the nontrivial nature of the Clifford algebra definition. In order to facilitate this proof, $\mathbb{Z}_2$-graded algebras are first introduced in the following section.

5.1 $\mathbb{Z}_2$-Graded Algebras

Let A be a real, unital algebra for which there exists vector subspaces A_0 and A_1 such that $A = A_0 \oplus A_1$, and for all x in A_i and y in A_j, the product xy is in $A_{(i+j) \bmod 2}$. Then A is referred to as a unital $\mathbb{Z}_2$-graded algebra.

If A and B are two $\mathbb{Z}_2$-graded algebras, it is possible to define a new $\mathbb{Z}_2$-graded algebra $A\widehat{\otimes}B$ as a certain product of A and B. This new algebra $A\widehat{\otimes}B$ is defined to have the structure

$$A\widehat{\otimes}B = (A_0 \otimes B_0) \;\oplus\; (A_1 \otimes B_1) \;\oplus\; (A_0 \otimes B_1) \;\oplus\; (A_1 \otimes B_0)$$

and a product following the rule that for $a_i \in A_i$, $a'_j \in A_j$, $b_r \in B_r$, and $b'_s \in B_s$, the product

$$(a_i \otimes b_r)(a'_j \otimes b'_s) = (-1)^{rj}(a_i a'_j) \otimes (b_r b'_s)\,.$$

Additionally, for $x, y, z \in A\widehat{\otimes}B$, multiplication distributes over addition so that $x(y+z) = xy + xz$ and $(x+y)z = xz + yz$. With this product, $A\widehat{\otimes}B$ can be decomposed into a direct sum of two vector subspaces $(A\widehat{\otimes}B)_0$ and $(A\widehat{\otimes}B)_1$, where

$$A\widehat{\otimes}B = \underbrace{\big[(A_0 \otimes B_0) \ \oplus \ (A_1 \otimes B_1)\big]}_{(A\widehat{\otimes}B)_0} \ \oplus \ \underbrace{\big[(A_0 \otimes B_1) \ \oplus \ (A_1 \otimes B_0)\big]}_{(A\widehat{\otimes}B)_1}$$

and it is seen that for two elements $x \in (A\widehat{\otimes}B)_i$ and $y \in (A\widehat{\otimes}B)_j$, the product $xy \in (A\widehat{\otimes}B)_{(i+j) \bmod 2}$. This is exactly the rule given above for the product on a $\mathbb{Z}_2$-graded algebra. In order for $A\widehat{\otimes}B$ to actually be a $\mathbb{Z}_2$-graded algebra it must also have an identity element and be associative with respect to multipliction. These two properties are now demonstrated.

If 1_A and 1_B are the identity elements of A and B, respectively, then $1_A \otimes 1_B$ is the identity element of $A\widehat{\otimes}B$. For $a_i \otimes b_r \in A_i \otimes B_r$, $a'_j \otimes b'_s \in A_j \otimes B_s$, and $a''_k \otimes b''_t \in A_k \otimes B_t$,

$$(a_i \otimes b_r)\big[(a'_j \otimes b'_s)(a''_k \otimes b''_t)\big] = (-1)^{sk}(-1)^{r(j+k \bmod 2)}(a_i a'_j a''_k \otimes b_r b'_s b''_t) \quad \text{and}$$

$$\big[(a_i \otimes b_r)(a'_j \otimes b'_s)\big](a''_k \otimes b''_t) = (-1)^{rj}(-1)^{k(r+s \bmod 2)}(a_i a'_j a''_k \otimes b_r b'_s b''_t)\,.$$

Associativity follows from $(-1)^{n \bmod 2} = (-1)^n$ since

$$(-1)^{rj}(-1)^{k(r+s \bmod 2)} = (-1)^{rj+kr+ks} = (-1)^{sk}(-1)^{r(j+k)} = (-1)^{sk}(-1)^{r(j+k \bmod 2)}\,.$$

Thus, $A\widehat{\otimes}B$ is indeed a $\mathbb{Z}_2$-graded algebra.

5.2 The Clifford Algebra of a Real Vector Space

There are several equivalent ways of defining the Clifford algebra [Lou01], but our definition will make use of a universal property stated in the next section.

5.2.1 The Universal Property of the Clifford Algebra

Let V be a real, finite-dimensional vector space on which is defined a nondegenerate quadratic form Q. Next, define the following sets:

$$A = V,$$

$$\mathcal{S} = \{W \mid W \text{ a real unital algebra with unit } 1_W\},$$

$$\mathcal{F}_{\mathcal{C}\ell} = \{f : V \to W \mid W \in \mathcal{S}\,, f \text{ linear, and } f(v)f(v) = Q(v)1_W\}, \text{ and}$$

$$\mathcal{H} = \{\phi \mid \phi \text{ is an algebra homomorphism}\}.$$

Relative to these four sets, if $\big(\mathcal{C}\ell(V,Q),\, f\big)$ is a universal pair then $\mathcal{C}\ell(V,Q)$ defines the Clifford algebra associated with vector space V having quadratic form Q. Thus, the universal function $f \in \mathcal{F}_{\mathcal{C}\ell}$ maps from V to $\mathcal{C}\ell(V,Q)$, and for any generic function $g : V \to W$ in $\mathcal{F}_{\mathcal{C}\ell}$, there exists a unique algebra homomorphism $\phi : \mathcal{C}\ell(V,Q) \to W$ in $\mathcal{H}$ such that $g = \phi \circ f$. The commuting diagram in Figure 5.1 illustrates the relationships that define the universal property of the Clifford algebra.

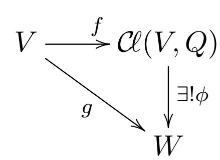

Figure 5.1: A commuting diagram depicting the universal property of the Clifford algebra $\mathcal{C}\ell(V,Q)$ associated with the real, finite-dimensional vector space V having nondegenerate quadratic form Q. By the universal property, the universal function $f \in \mathcal{F}_{\mathcal{C}\ell}$ is the function such that for any generic pair (W, g) there exists a unique algebra homomorphism ϕ such that $g = \phi \circ f$.

5.2.2 Existence and Nontriviality of the Clifford Algebra

With the definition in place, we next turn to verifying that such a universal pair does in fact exist. This is achieved in a direct manner by constructing one as a quotient algebra of the tensor algebra. We proceed with this construction in the next theorem.

Theorem 5.1. Let $T(V)$ be the tensor algebra, and let $i : V \to T(V)$ be the inclusion map. Let I be the ideal generated by all elements of $T(V)$ of the form $i(v) \otimes i(v) - Q(v)1_{T(V)}$. Take $\pi : T(V) \to T(V)/I$ to be the canonical projection. Then, $\big(T(V)/I,\, \pi \circ i\big)$ is a universal pair for the universal property of Clifford algebras. Consequently, $\mathcal{C}\ell(V, Q) = T(V)/I$.

Proof. From the definition of the ideal and the fact that π is a homomorphism it follows that $\pi \circ i \in \mathcal{F}_{\mathcal{C}\ell}$. Any function $g \in \mathcal{F}_{\mathcal{C}\ell}$ is also a member of $\mathcal{F}_{T(V)}$ used in the definition of the tensor algebra's universal property (Section 4.3.2). Thus, there exists a unique algebra homomorphism $\phi' : T(V) \to W$ such that $g = \phi' \circ i$. The generators of I are in $\ker \phi'$ since

$$\phi'\big(i(v) \otimes i(v) - Q(v)1_{T(V)}\big) = \phi'\big(i(v)\big)\phi'\big(i(v)\big) - Q(v)1_W = g(v)^2 - Q(v)1_W = 0\,.$$

However, $\ker \phi'$ is itself an ideal, thus $I \subseteq \ker \phi'$. By Theorem 1.7, there exists a unique algebra homomorphism $\phi : T(V)/I \to W$, such that $\phi' = \phi \circ \pi$. These relations are illustrated in the commuting diagram of Figure 5.2. Thus, $\big(T(V)/I,\, \pi \circ i\big)$ is a universal pair for the universal property of Clifford algebras and $\mathcal{C}\ell(V, Q) = T(V)/I$. □

We next turn to proving that the Clifford algebra is nontrivial. This will take quite a bit of work to show.

Lemma 5.2. If $\dim V = 1$ then $\mathcal{C}\ell(V, Q)$ is nontrivial.

Proof. Take any basis of V and let e be the single element comprising the basis. Any vector $v \in V$ can be written as $v = re$ for some $r \in \mathbb{R}$. Without loss of generality, suppose Q

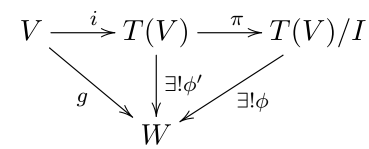

Figure 5.2: The commuting diagram depicts the relationships used in the proof of existence for Clifford algebra $\mathcal{C}\ell(V,Q)$. The proof makes use of the tensor algebra's universal property to show that $\big(T(V)/I,\, \pi \circ i\big)$ is a universal pair with respect to the universal property of Clifford algebras. Theorem 1.7 is used to obtain the unique algebra homomorphism ϕ.

has signature $(1,0,0)$. Referring to Section 4.3.4, the tensor algebra $T(V)$ has the basis $\mathscr{B} = \{1,\ e,\ e \otimes e,\ e \otimes e \otimes e,\ \ldots\}$.

Define a map $f : \mathscr{B} \to \mathbb{R}[x]$, where $\mathbb{R}[x]$ is the ring of polynomials over $\mathbb{R}$ of the single indeterminate x, such that

$$f(\underbrace{e \otimes \cdots \otimes e}_{m \text{ times}}) = x^m$$

where $m \geq 0$ and $m = 0$ corresponds to $f(1_{T(V)}) = 1$. Then $f(\mathscr{B}) = \{1, x, x^2, x^3, \ldots\}$ which is a basis for $\mathbb{R}[x]$. The function f extended linearly and as a homomorphism to all of $T(V)$ makes f an isomorphism between the algebras $T(V)$ and $\mathbb{R}[x]$. Under f, the ideal $I = \langle v \otimes v - Q(v)1_{T(V)} \mid v \in V \rangle$ maps to

$$f(I) = \big\langle f\big(v \otimes v - Q(v)1_{T(V)}\big) \;\big|\; v \in V \big\rangle = \langle f(re)f(re) - Q(re) \mid r \in \mathbb{R} \rangle$$

$$= \langle x^2 - 1 \rangle\,.$$

Therefore, it is enough to show that ideal $\langle x^2 - 1 \rangle$ is not all of $\mathbb{R}[x]$. Due to the commutativity of $\mathbb{R}[x]$, the ideal is equal to $\{(x^2-1)p(x) \mid p \in \mathbb{R}[x]\}$. Every element of $\langle x^2 - 1 \rangle$ has 1 as a root; however, this is not the case for $\mathbb{R}[x]$ and so $\langle x^2 - 1 \rangle \neq \mathbb{R}[x]$. □

Lemma 5.3. 1. There exists a unique vector space automorphism t of $\mathcal{C}\ell(V,Q)$ such that for all $v \in V$ and $x, y \in \mathcal{C}\ell(V,Q)$,

(a) $t(xy) = t(y)t(x)$,

(b) $t\big(\pi(v)\big) = \pi(v)$, and

(c) $t \circ t = \mathrm{Id}$, the identity map.

2. There exists a unique algebra automorphism α of $\mathcal{C}\ell(V,Q)$ such that for all $v \in V$,

(a) $\alpha\big(\pi(v)\big) = -\pi(v)$, and

(b) $\alpha \circ \alpha = \mathrm{Id}$.

Proof of 1. The universal property of the Clifford algebra will be used to demonstrate that t exists and is unique. For this proof we make a distinction between the set of elements $\mathrm{CL}(V,Q)$ in a Clifford algebra (i.e., neglecting the binary operations), and the Clifford algebra itself (with the binary operations): $\mathcal{C}\ell(V,Q) = \big(\mathrm{CL}(V,Q), +, \cdot, *\big)$. In addition to $\mathcal{C}\ell(V,Q)$, we introduce an algebra $\mathcal{C}\ell^{op}(V,Q) = (\mathrm{CL}(V,Q), +, \cdot, \star)$, which is defined on the same set, and shares the same addition and scalar multiplication operations as $\mathcal{C}\ell(V,Q)$, but which has a different multiplication rule. Multiplication in $\mathcal{C}\ell^{op}(V,Q)$, is defined as follows:

$$x \star y = y * x \quad \text{for all } x, y \in \mathrm{CL}(V,Q)\,.$$

The conscientious reader may verify that $\star$ is associative and distributes over addition, and that $1_{\mathcal{C}\ell} \in \mathrm{CL}(V,Q)$ is the unit in $\mathcal{C}\ell^{op}(V,Q)$, making $\mathcal{C}\ell^{op}(V,Q)$ a real unital algebra. As such, $\mathcal{C}\ell^{op}(V,Q)$ is a generic set relative to the universal property for Clifford algebras.

At this point we define a map $\pi^{op} : V \to \mathcal{C}\ell^{op}(V,Q)$ such that $\pi^{op}(v) = \pi(v)$ for all $v \in V$. As they are defined, $\mathcal{C}\ell(V,Q)$ and $\mathcal{C}\ell^{op}(V,Q)$ are identical as vector spaces when ignoring their algebra multiplication operations, and this implies that π^{op} is linear. From the multiplication rule of $\mathcal{C}\ell^{op}(V,Q)$ it follows that

$$\pi^{op}(v) \star \pi^{op}(v) = \pi(v) * \pi(v) = Q(v) 1_{\mathcal{C}\ell}\,.$$

By construction, π^{op} is a generic function relative to the Clifford algebra's universal property. Therefore, there is a unique algebra homomorphism $t : \mathcal{C}\ell(V,Q) \to \mathcal{C}\ell^{op}(V,Q)$ such that $\pi^{op} = t \circ \pi$. Properties 1a and 1b follow immediately.

To demonsrate property 1c, apply $t \circ t$ to the product $\prod_{i=1}^{n} \pi(v_i)$, where the $v_i \in V$ are arbitrary. Application of the first t gives

$$\begin{aligned} t\Big(\prod_{i=1}^{n} \pi(v_i)\Big) &= t\big(\pi(v_1)\big) \star t\big(\pi(v_2)\big) \star \cdots \star t\big(\pi(v_{n-1})\big) \star t\big(\pi(v_n)\big) \\ &= \pi^{op}(v_1) \star \pi^{op}(v_2) \star \cdots \star \pi^{op}(v_{n-1}) \star \pi^{op}(v_n) \\ &= \pi(v_n) * \pi(v_{n-1}) * \cdots * \pi(v_2) * \pi(v_1)\,, \end{aligned}$$

which simply reverses the order of the original product. Similarly, application of the second t reverses the order yet again, yielding the original product. It follows from our description of the basis for the tensor algebra in Section 4.3.4 that $t \circ t$ is the identity. □

Proof of 2. Consider $\mathcal{C}\ell(V,Q)$ with the usual product and select $\big(\mathcal{C}\ell(V,Q), -\pi\big)$ as the generic pair. Note that $-\pi$ is a valid generic function since for all $v \in V$, the product $\big(-\pi(v)\big)^2 = \pi^2(v) = Q(v)1_{\mathcal{C}\ell}$. By the universal property, there exists a unique algebra homomorphism α such that $\alpha \circ \pi = -\pi$. Therefore, for all $v \in V$,

$$\alpha\big(\pi(v)\big) = -\pi(v)\,, \quad \text{and}$$

$$(\alpha \circ \alpha)\big(\pi(v)\big) = \alpha\big(-\pi(v)\big) = -\alpha\big(\pi(v)\big) = \pi(v)\,.$$

□

Corollary 5.4 (Clifford algebras are $\mathbb{Z}_2$-graded algebras.)**.** Let

$$\mathcal{C}\ell(V,Q)_0 = \{x \in \mathcal{C}\ell(V,Q) \mid \alpha(x) = x\} \quad \text{and}$$

$$\mathcal{C}\ell(V,Q)_1 = \{x \in \mathcal{C}\ell(V,Q) \mid \alpha(x) = -x\}\,.$$

These are easily verified to be vector subspaces of $\mathcal{C}\ell(V,Q)$. Then, $\mathcal{C}\ell(V,Q)$ has the following properties:

1. $\mathcal{C}\ell(V,Q) = \mathcal{C}\ell(V,Q)_0 \oplus \mathcal{C}\ell(V,Q)_1$,
2. for all $x \in \mathcal{C}\ell(V,Q)_i$ and $y \in \mathcal{C}\ell(V,Q)_j$, $xy \in \mathcal{C}\ell(V,Q)_k$, where $k = (i+j) \bmod 2$, and
3. we have $1_{\mathcal{C}\ell} \in \mathcal{C}\ell(V,Q)_0$ and for each $v \in V$, $\pi(v) \in \mathcal{C}\ell(V,Q)_1$.

Consequently, $\mathcal{C}\ell(V,Q)$ is a $\mathbb{Z}_2$-graded algebra.

Proof. 1. If there is an element x common to both subspaces, then

$$\alpha(x) = -\alpha(x) \quad \Rightarrow \quad \alpha(x) = 0\,.$$

Since α is an isomorphism, this means $x = 0$, and therefore

$$\mathcal{C}\ell(V,Q)_0 \cap \mathcal{C}\ell(V,Q)_1 = \{0\}\,.$$

If $\mathscr{B}$ is the basis of the tensor algebra given in Section 4.3.4, then the image $\pi(\mathscr{B})$ is a generating set for $\mathcal{C}\ell(V,Q)$. An element $\tau \in \mathscr{B}$ has the general form $\tau = e_{i_1} \otimes \cdots \otimes e_{i_p}$ and $\pi(\tau) = \pi(e_{i_1}) \cdots \pi(e_{i_p})$ is the general form of elements in $\pi(\mathscr{B})$. Then

$$\alpha\big(\pi(e_{i_1}) \cdots \pi(e_{i_p})\big) = \alpha\big(\pi(e_{i_1})\big) \cdots \alpha\big(\pi(e_{i_p})\big) = (-1)^p \pi(e_{i_1}) \cdots \pi(e_{i_p})\,. \tag{5.1}$$

Thus, if p is even, $\pi(\tau) \in \mathcal{C}\ell(V,Q)_0$ and if p is odd then $\pi(\tau) \in \mathcal{C}\ell(V,Q)_1$. Therefore, $\mathcal{C}\ell(V,Q) = \mathcal{C}\ell(V,Q)_0 \oplus \mathcal{C}\ell(V,Q)_1$.

2. Proof of property 2 is shown directly by examining each case. We prove only the case for $i = j = 0$, the other cases being similar. For elements $x, y \in \mathcal{C}\ell(V,Q)_0$ we have $\alpha(xy) = \alpha(x)\alpha(y) = xy \in \mathcal{C}\ell(V,Q)_0$, proving the result in this case.

3. Since α is a homomorphism, $\alpha(1_{\mathcal{C}\ell}) = 1_{\mathcal{C}\ell}$. That $\pi(v)$ is an element of $\mathcal{C}\ell(V,Q)_1$ follows from Lemma 5.3.

□

Theorem 5.5. The Clifford algebra $\mathcal{C}\ell(V,Q)$ is nontrivial.

Proof. It will first be shown that $\mathcal{C}\ell(V,Q)$ is nontrivial if there exists a nontrivial generic pair. Having done this, a generic pair will be constructed that will be shown to be nontrivial by induction on $\dim V$.

To begin, suppose a nontrivial generic pair (W, g) exists. Then there is an algebra homomorphism ϕ such that for each $v \in V$, $\phi\big(\pi\big(i(v)\big)\big) = g(v)$. Since $g(v)g(v) = Q(v)1_W$, and this is not zero for at least one v, it follows that $\pi\big(i(v)\big)$ is not identically zero, and so V is not contained in the ideal I. Thus, the quotient algebra $T(V)/I$ is nontrivial.

By Lemma 5.2, $\mathcal{C}\ell(V,Q)$ is nontrivial when $\dim V = 1$. Assume $\mathcal{C}\ell(V,Q)$ is nontrivial for $\dim V \leq k$ and let $\dim V = k+1$.

Take $\{e_1, \ldots, e_{k+1}\}$ to be a basis for V which is orthogonal relative to the quadratic form's associated bilinear form B_Q. Let V_1 be the subspace generated by e_1; let V_2 be the subspace generated by $\{e_2, \ldots, e_{k+1}\}$. The restriction $Q_i = Q\big|_{V_i}$ remains a quadratic form. Note that it is possible to form the Clifford algebras $\mathcal{C}\ell(V_i, Q_i)$ from the subspaces V_i. By the inductive hypothesis, each $\mathcal{C}\ell(V_i, Q_i)$ is nontrivial.

The vector space $V \cong V_1 \oplus V_2$ so every $v \in V$ can be written uniquely as $v_1 + v_2 \in V_1 \oplus V_2$, where $v_i \in V_i$. By orthogonality of v_1 and v_2,

$$Q(v_1 + v_2) = Q(v_1) + Q(v_2) = Q_1(v_1) + Q_2(v_2)\,.$$

From Corollary 5.4, each $\mathcal{C}\ell(V_i, Q_i)$ is a $\mathbb{Z}_2$-graded algebra and so we can form the product

$\mathcal{C}\ell(V_1, Q_1) \mathbin{\widehat{\otimes}} \mathcal{C}\ell(V_2, Q_2)$ defined in Section 5.1.

Define a linear function $j : V_1 \oplus V_2 \to \mathcal{C}\ell(V_1, Q_1) \mathbin{\widehat{\otimes}} \mathcal{C}\ell(V_2, Q_2)$ by $j(v_1, v_2) = \pi(v_1) \otimes 1 + 1 \otimes \pi(v_2)$. Recalling the product defined in Section 5.1 and the fact that $\pi(v_i) \in \mathcal{C}\ell(V_i, Q_i)_1$ and $1 \in \mathcal{C}\ell(V_i, Q_i)_0$, we obtain

$$\begin{aligned} j(v_1, v_2)j(v_1, v_2) &= \left[\pi(v_1) \otimes 1 \ + \ 1 \otimes \pi(v_2)\right] \left[\pi(v_1) \otimes 1 \ + \ 1 \otimes \pi(v_2)\right] \\ &= \pi(v_1)^2 \otimes 1 \ + \ 1 \otimes \pi(v_2)^2 \ + \ \pi(v_1) \otimes \pi(v_2) \ - \ \pi(v_1) \otimes \pi(v_2) \\ &= Q_1(v_1) 1 \otimes 1 \ + \ Q_2(v_2) 1 \otimes 1 = Q(v_1 + v_2) 1 \,. \end{aligned}$$

Therefore, $\left(\mathcal{C}\ell(V_1, Q_1) \mathbin{\widehat{\otimes}} \mathcal{C}\ell(V_2, Q_2),\, j\right)$ is a nontrivial generic pair. □

5.2.3 A Basis for the Clifford Algebra

In this section $\{e_1, \ldots, e_n\}$ is a basis for V which is orthogonal with respect to the quadratic form's associated bilinear form B_Q.

Lemma 5.6. Let $\tau = e_{j_1} \otimes e_{j_2} \otimes \cdots \otimes e_{j_p}$ in which factors are distinct. Let σ be a permutation of p elements and $\lambda_\sigma(\tau) = e_{j_{\sigma(1)}} \otimes \cdots \otimes e_{j_{\sigma(p)}}$. Then $\operatorname{sgn}(\sigma)\lambda_\sigma(\tau) - \tau$ is an element of I.

Proof. Consider a vector v of the form $v = e_i + e_j$. Then,

$$\begin{aligned} v \otimes v - Q(v) 1_{T(V)} &= (e_i + e_j) \otimes (e_i + e_j) - B_Q(e_i + e_j,\, e_i + e_j) 1_{T(V)} \\ &= e_i \otimes e_i \ + \ e_j \otimes e_j \ + \ e_i \otimes e_j \ + \ e_j \otimes e_i \\ &\quad - B_Q(e_i,\, e_i) - B_Q(e_j,\, e_j) - 2B_Q(e_i,\, e_j) \\ &= e_i \otimes e_j + e_j \otimes e_i + e_i \otimes e_i - Q(e_i) 1_{T(V)} + e_j \otimes e_j - Q(e_j) 1_{T(V)} \,, \end{aligned}$$

where we have made use of the orthogonality of e_i and e_j. From this it is seen that

$$e_i \otimes e_j \ + \ e_j \otimes e_i \in I. \tag{5.2}$$

The desired result then follows using the same reasoning as in Section 4.4.2 where it was shown that $J_2 \subseteq J_1$. $\square$

Let $\mathscr{B}$ be the set of all basis elements of $T(V)$ and $\mathscr{B}'$ the set of basis elements with no repeated factor and factor indices in strictly increasing order. We claim that every element in $\mathscr{B}$ differs from a certain multiple of an element in $\mathscr{B}'$ by an element of the ideal. Lemma 5.6 shows this to be true for elements with no repeated factor. For basis elements with at least one repeated factor, we argue by induction on the grade of the basis element. Consider such a basis element with the factor e_i appearing twice. By Equation 5.2, the basis element can be rewritten, up to a factor of ± 1 and an element of the ideal, with both factors of e_i appearing on the left. But $e_i \otimes e_i$ is equal to $Q(e) 1_{T(V)}$ plus an element of the ideal. Therefore, the reordered basis element can be rewritten as a scalar multiple of a lower grade basis element, up to an element of the ideal. The result then follows by induction. This completes the proof of the claim.

Theorem 5.7. The set $\mathscr{B}_{\mathcal{C}\ell} = \{\pi(e_{i_1}) \cdots \pi(e_{i_p}) \mid 1 \leq i_1 < \cdots < i_p \leq n\,,\ 0 \leq p \leq n\}$ is a basis of $T(V)/I$.

Proof. It must be shown that $\mathscr{B}_{\mathcal{C}\ell}$ spans $\mathcal{C}\ell(V, Q)$ and is linearly independent. Since π is surjective, it maps a basis of $T(V)$ to a spanning set of $\mathcal{C}\ell(V, Q)$. However, it was shown in the preceding discussion that, under π, every basis element of $T(V)$ maps to $\mathscr{B}_{\mathcal{C}\ell}$. Thus, $\mathscr{B}_{\mathcal{C}\ell}$ spans $\mathcal{C}\ell(V, Q)$.

We now prove linear independence. Suppose we have

$$\sum_{k=1}^{m} \left(\beta_k \prod_{j=1}^{p_k} \pi(e_{i_{k,j}}) \right) = 0\,, \tag{5.3}$$

where if $p_k \neq 0$ then the indices satisfy $1 \leq i_{k,1} < \cdots < i_{k,p_k} \leq n$, and if $p_k = 0$ then the term is defined to be $\beta_k 1_{\mathcal{C}\ell}$. We must prove that $\beta_1 = \cdots = \beta_m = 0$.

The automorphisms α, and t, defined in Lemma 5.3 will be used. Apply α to both sides of Equation (5.3). In general, the k^{th} term of the resulting sum will contain

$$\alpha\big(\pi(e_{i_{k,1}})\cdots\pi(e_{i_{k,p_k}})\big) = (-1)^{p_k}\pi(e_{i_{k,1}})\cdots\pi(e_{i_{k,p_k}})\,.$$

Let the number p_k of factors in a term be called the *length* of the term. Terms with an even (odd) number of such factors are said to have even (odd) length. Thus, terms of even length are invariant under α and terms of odd length change sign. Therefore,

$$\alpha\left(\sum_{k=1}^{m}\left(\beta_k\prod_{j=1}^{p_k}\pi(e_{i_{k,j}})\right)\right) = \sum_{\substack{k=1\\p_k\text{ even}}}^{m}\left(\beta_k\prod_{j=1}^{p_k}\pi(e_{i_{k,j}})\right) - \sum_{\substack{k=1\\p_k\text{ odd}}}^{m}\left(\beta_k\prod_{j=1}^{p_k}\pi(e_{i_{k,j}})\right)$$

$$= 0$$

$$= \sum_{\substack{k=1\\p_k\text{ even}}}^{m}\left(\beta_k\prod_{j=1}^{p_k}\pi(e_{i_{k,j}})\right) + \sum_{\substack{k=1\\p_k\text{ odd}}}^{m}\left(\beta_k\prod_{j=1}^{p_k}\pi(e_{i_{k,j}})\right).$$

This implies that the terms of odd length sum to zero, as do the terms of even length, i.e.,

$$\sum_{\substack{k=1\\p_k\text{ odd}}}^{m}\left(\beta_k\prod_{j=1}^{p_k}\pi(e_{i_{k,j}})\right) = 0$$

and

$$\sum_{\substack{k=1\\p_k\text{ even}}}^{m}\left(\beta_k\prod_{j=1}^{p_k}\pi(e_{i_{k,j}})\right) = 0\,.$$

Now apply t to both sides of the original linear combination. Without loss of generality, we can consider the terms of odd length separately from the terms of even length, since we know that each of these collections sums to zero separately. First, examine the even length terms; the effect of t on any single term is

$$t\big(\beta_k \pi(e_{i_{k,1}}) \cdots \pi(e_{i_{k,p_k}})\big) = \beta_k \pi(e_{i_{k,p_k}}) \cdots \pi(e_{i_{k,1}}) \tag{5.4}$$

From Lemma 5.6, for any $e \in \mathscr{B}$

$$\operatorname{sgn}(\sigma)\pi(\lambda_\sigma e) = \pi(e)\,,$$

which implies that the right-hand side of Equation (5.4) (now with factors in reversed order) can be re-written with the factors in the original order multiplied by ± 1 depending on the sign of the permutation σ needed to regain the correct ordering. Equivalently, the sign change is based on the number of transpositions required to correctly order the factors. For terms of length $p_k = 0 \pmod 4$ an even number of transpositions correctly orders the factors, while for terms of length $p_k = 2 \pmod 4$ an odd number of transpositions correctly orders the factors. Thus, terms of length $p_k = 2 \pmod 4$ gain a factor of -1 and we have

$$\begin{aligned} t\left(\sum_{\substack{k=1 \\ p_k \text{ even}}}^{m} \left(\beta_k \prod_{j=1}^{p_k} \pi(e_{i_{k,j}})\right)\right) &= \sum_{\substack{k=1 \\ p_k \bmod 4=0}}^{m} \left(\beta_k \prod_{j=1}^{p_k} \pi(e_{i_{k,j}})\right) - \sum_{\substack{k=1 \\ p_k \bmod 4=2}}^{m} \left(\beta_k \prod_{j=1}^{p_k} \pi(e_{i_{k,j}})\right) \\ &= 0 \\ &= \sum_{\substack{k=1 \\ p_k \bmod 4=0}}^{m} \left(\beta_k \prod_{j=1}^{p_k} \pi(e_{i_{k,j}})\right) + \sum_{\substack{k=1 \\ p_k \bmod 4=2}}^{m} \left(\beta_k \prod_{j=1}^{p_k} \pi(e_{i_{k,j}})\right) . \end{aligned}$$

This implies that the terms of length $p_k = 0 \pmod 4$ must sum to zero, and the terms of length $p_k = 2 \pmod 4$ must also sum to zero. A similar analysis of the terms of odd length yields that those terms of length $p_k = 1 \pmod 4$ must sum to zero, as must the terms of length $p_k = 3 \pmod 4$. Let $\sum_i$ be the sum of the terms of length $i \pmod 4$.

Consider each $\sum_i = 0$ individually. If the sum only has one term, then automatically its coefficient must be zero. If there is more than one term, then there is a $\pi(e_\ell)$ which is a factor of one of these terms, but not of every term in the sum. Multiply the equation by this factor:

$$\pi(e_\ell) \sum_i = 0 \,.$$

Upon multiplication, each term that previously contained $\pi(e_\ell)$ will shorten in length by 1 and gain a factor of $\pm Q(e_\ell) 1_{\mathcal{C}\ell}$. Those that did not already contain $\pi(e_\ell)$ will increase in length by 1 and gain a factor of ± 1. The product $\pi(e_\ell) \sum_i$ is now a sum of terms of length $i \pm 1 \pmod 4$. Now it is possible to apply t to both sides of the equation and again achieve the result that the terms of length $i+1 \pmod 4$ must sum to zero and those of length $i-1 \pmod 4$ must also sum to zero. Once again, if either of these new sums consists of a single term, then its coefficient β_k must be zero. While the term may have accumulated a factor $\pm Q(e_\ell)$, the quadratic form is nondegenerate and so $Q(e_\ell)$ is non-zero, thus ensuring that β_k is indeed zero. If either sum has more than one term we continue repeating this process and at each step the sums become shorter until we have deduced that each coefficient must be zero. □

Corollary 5.8. Clifford algebra $\mathcal{C}\ell(V, Q)$ and exterior algebra $\Lambda(V)$ are isomorphic as vector spaces.

Proof. This follows immediately using the basis for the exterior algebra demonstrated in Section 4.4.4 and Theorem 5.7. Both have dimension 2^n where n is the dimension of V. □

Chapter 6: Classification of the Real Clifford Algebras

Each real Clifford algebra is isomorphic to a matrix algebra or the two-fold direct sum of a matrix algebra with itself. Although all these matrix algebras are over the field of reals, they take their entries from the real numbers, the complex numbers or the quaternions. The classification of the Clifford algebras is according to these matrix algebras.

In this chapter, the necessary isomorphisms are developed so that, given any real Clifford algebra, the isomorphic matrix algebra can be determined. The program we will follow is to first classify each of the lowest dimensional Clifford algebras (those generated from vector spaces having dimension equal to 1 or 2) by demonstrating its isomorphic matrix algebra. The classification will then proceed to higher dimensional Clifford algebras by decomposing them into tensor products of the lowest dimensional Clifford algebras, and then showing how these tensor products can be collapsed to a single matrix algebra or a direct-sum of two matrix algebras.

Our discussion of Clifford algebras has applied to those defined with a finite dimensional vector space V over the field of reals accompanied by a nondegenerate quadratic form Q. Up to this point the properties of Clifford algebras that were developed have been independent of the particular quadratic form. In fact, it is the quadratic form that determines the algebraic structure of the Clifford algebra and this point will come to the forefront in the classification.

We know that the symmetric bilinear form of every quadratic form on V having signature $(p, m, 0)$ can be diagonalized by a change of basis to have p plus ones and m minus ones (Theorem 3.3, Sylvester's Law of Inertia). Furthermore, all real vector spaces with a given dimension $n = p + m$ are isomorphic. Thus, the signature and the dimension are the properties of Q and V that affect the structure of Clifford algebra $\mathcal{C}\ell(V, Q)$. For these reasons, we discontinue the notation $\mathcal{C}\ell(V,Q)$ and instead use the notation $\mathcal{C}\ell_{p,m}$. This notation highlights the influence that the signature of the quadratic form has on the structure of the Clifford algebra, and in fact it is the means by which we classify the Clifford algebras. Other than its dimension, the particular vector space is unimportant. For our purposes, we will assume in this chapter that the underlying vector space is $\mathbb{R}^{p+m}$.

The classification of the lowest dimensional Clifford algebras is covered in the first two sections of the chapter. The $p=0$ cases, that is, $\mathcal{C}\ell_{0,1}$ and $\mathcal{C}\ell_{0,2}$, are classified in Section 6.1. The $m=0$ cases and the case $p=m=1$ are discussed in Section 6.2.

6.1 Algebras of the Complex Numbers and Quaternions

The algebras of the complex numbers and the quaternions, both over the field of reals, are in fact Clifford algebras and make their way into our classification as $\mathcal{C}\ell_{0,1}$ and $\mathcal{C}\ell_{0,2}$, respectively. The proofs of isomorphism are similar and make use of the respective Clifford algebra's universal property.

Proposition 6.1. The algebra $\mathbb{C}$ over the field of reals is isomorphic to $\mathcal{C}\ell_{0,1}$.

Proof. Let $\{1_{\mathcal{C}\ell}, e\}$, where $e=\pi(1)$, be the orthonormal basis for $\mathcal{C}\ell_{0,1}$ that was demonstrated in Theorem 5.7. As a real algebra, $\mathbb{C}$ has $\{1,i\}$ as a basis. Define the map $g:\mathbb{R}\to\mathbb{C}$ by $g(1)=i$ with linear extension. Note that for any $r\in\mathbb{R}$, $g^2(r)=r^2\cdot i^2=-r^2=Q(r)\cdot 1$. Therefore, $(\mathbb{C},g)$ is a generic pair and by the universal property of $\mathcal{C}\ell_{0,1}$ there exists an algebra homomorphism $\phi:\mathcal{C}\ell_{0,1}\to\mathbb{C}$ such that $g=\phi\circ\pi$. As a homomorphism, $\phi(1_{\mathcal{C}\ell})=1$. Furthermore,

$$\phi(e)=\phi(\pi(1))=g(1)=i\,.$$

Having shown that ϕ is bijective between the basis sets implies ϕ is bijective between the algebras, and thus ϕ is an algebra isomorphism. □

Having shown the complex numbers to be a Clifford algebra, it will next be shown that the quaternions are also a Clifford algebra. First, quaternion algebra is briefly reviewed.

6.1.1 An Overview of Quaternion Algebra

The quaternions comprise a four-dimensional, real unital algebra, with the standard basis denoted by $\{1, i, j, k\}$. In general, then, a quaternion has the form $q = a + bi + cj + dk$, for some $a, b, c, d \in \mathbb{R}$. Basis element 1 is the identity under quaternion multiplication, and the multiplication is defined on the other basis elements via the following relations:

$$i^2 = j^2 = k^2 = ijk = -1\,.$$

From these relations we can deduce that $ij = k$, $jk = i$, $ki = j$, and furthermore that i, j, and k *anticommute*, that is $ij = -ji$, $jk = -kj$, and $ki = -ik$.

Viewed as just a vector space, the quaternions are isomorphic to the direct sum $\mathbb{R} \oplus \mathbb{R}^3$. With this perspective, a general element can be written as $q = s + \vec{v}$, where $s \in \mathbb{R}$ and $\vec{v} \in \mathbb{R}^3$. We insist that quaternion $a + bi + cj + dk$ maps to $a + (b, c, d)$ in $\mathbb{R} \oplus \mathbb{R}^3$. It can be useful to think of s as the "scalar" part of the quaternion and $\vec{v}$ as the "vector" part. Then there is a useful identity for the multiplication of two quaternions. Multiplying $q_1 = s_1 + \vec{v}_1$ and $q_2 = s_2 + \vec{v}_2$ gives

$$q_1 q_2 = s_1 s_2 - \vec{v}_1 \cdot \vec{v}_2 + s_1 \vec{v}_2 + s_2 \vec{v}_1 + \vec{v}_1 \times \vec{v}_2\,,$$

where $\cdot$ and $\times$ are the standard dot product and cross product defined on $\mathbb{R}^3$.

With i, j, and k squaring to -1, quaternions can be thought of as a four-dimensional analog of complex numbers, having a real component and three imaginary components. Similar to complex numbers, we define a *conjugation* operation on the quaternions. For any quaternion $q = a + bi + cj + dk$, the *conjugate* of q is $\overline{q} = a - bi - cj - dk$. It then follows that for $q_1 = s_1 + \vec{v}_1$ and $q_2 = s_2 + \vec{v}_2$,

$$\overline{q}_1\,\overline{q}_2 = s_1 s_2 - \vec{v}_1 \cdot \vec{v}_2 - s_1\vec{v}_2 - s_2\vec{v}_1 - \vec{v}_2 \times \vec{v}_1 = \overline{(q_2 q_1)}\,.$$

Instead of viewing the quaternions as a four-dimensional real vector space, it is possible to interpret them as a two-dimensional complex vector space. The map that takes a quaternion $q = a + bi + cj + dk$ in $\mathbb{R}^4$ to (z_1, z_2) in $\mathbb{C}^2$, where $z_1 = a + bi$ and $z_2 = c + di$, is a bijection between the two spaces. This viewpoint will be used ahead in Proposition 6.7.

The quaternions are represented by the symbol $\mathbb{H}$ in honor of Sir William Rowan Hamilton who invented them.

Proposition 6.2. The quaternion algebra is isomorphic to $\mathcal{C}\ell_{0,2}$.

Proof. Let $\{1_{\mathcal{C}\ell}, e_1, e_2, e_{12}\}$ be the orthonormal basis for $\mathcal{C}\ell_{0,2}$ given in Theorem 5.7, where $e_1 = \pi(1,0)$ and $e_2 = \pi(0,1)$. The set $\{1, i, j, k\}$ is a basis for $\mathbb{H}$. Define the map $g : \mathbb{R}^2 \to \mathbb{H}$ by specifying that $g(1,0) = i$, $g(0,1) = j$ and extending the map linearly. For any $(x, y) \in \mathbb{R}^2$,

$$g^2(x,y) = (xi + yj)(xi + yj) = -x^2 - y^2 + xyk - yxk = -x^2 - y^2 = Q(x,y) \cdot 1\,.$$

Therefore, $(\mathbb{H}, g)$ is a generic pair and by the universal property of $\mathcal{C}\ell_{0,2}$ we obtain the algebra homomorphism $\phi : \mathcal{C}\ell_{0,2} \to \mathbb{H}$. This use of the universal property is shown schematically in Figure 6.1. Being a homomorphism, $\phi(1_{\mathcal{C}\ell}) = 1$ and furthermore

$$\phi(e_1) = \phi\big(\pi(1,0)\big) = g(1,0) = i\,,$$

$$\phi(e_2) = \phi\big(\pi(0,1)\big) = g(0,1) = j\,, \text{ and}$$

$$\phi(e_{12}) = \phi\big(\pi(1,0)\pi(0,1)\big) = g(1,0)g(0,1) = ij = k\,.$$

Having demonstrated ϕ to be bijective between the basis sets implies ϕ is an algebra isomorphism. $\square$

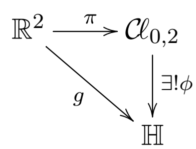

Figure 6.1: A commuting diagram showing the universal property of $\mathcal{C}\ell_{0,2}$ as used in Proposition 6.2 to deduce that there is an algebra homomorphism from $\mathcal{C}\ell_{0,2}$ to $\mathbb{H}$. The inclusion map from $\mathbb{R}^2$ to the tensor algebra $T(\mathbb{R}^2)$ has been suppressed.

6.2 Algebras of the Split-Complex Numbers and $\mathbb{R}(2)$

Before the discussion of the split-complex numbers, it is necessary to cover the algebra structure of a direct sum of algebras. Given two algebras A_1 and A_2, the direct sum $A_1 \oplus A_2$ has elements that are pairs of the form (a_1, a_2), where $a_1 \in A_1$ and $a_2 \in A_2$. The vector space structure of $A_1 \oplus A_2$ is the same as that of a direct sum of vector spaces. The algebra multiplication in $A_1 \oplus A_2$ is defined component-wise by $(a_1, a_2)(a_1', a_2') = (a_1 a_1', a_2 a_2')$. Thus, the multiplication in $A_1 \oplus A_2$ inherits distributivity and associativity from the algebra muliplication operations of A_1 and A_2, and the unit on $A_1 \oplus A_2$ is $1_{A_1} \oplus 1_{A_2}$.

Proposition 6.3. The following algebras are isomorphic:

1. $\mathcal{C}\ell_{1,0} \cong \mathbb{R} \oplus \mathbb{R}$ (called the split-complex numbers), and

2. $\mathcal{C}\ell_{2,0} \cong \mathcal{C}\ell_{1,1} \cong \mathbb{R}(2)$.

Proof. As with Propositions 6.1 and 6.2, these three isomorphisms will be proved using the universal property of the corresponding Clifford algebra.

Proof of 1. Let $\{1_{\mathcal{Cl}}\,,\ e\}$ be the standard basis for $\mathcal{Cl}_{1,0}$ with $e = \pi(1)$. The set $\big\{(1,1),\ (1,-1)\big\}$ is a basis for $\mathbb{R}\oplus\mathbb{R}$. Define the map $g : \mathbb{R} \to \mathbb{R}\oplus\mathbb{R}$ by $g(1) = (1,-1)$ and extending linearly to all of $\mathbb{R}$. Then, for any $r \in \mathbb{R}$,

$$g^2(r) = (r,-r)(r,-r) = (r^2,r^2) = r^2 \cdot (1,1) = Q(r) \cdot (1,1)\,.$$

Hence, $(\mathbb{R}\oplus\mathbb{R}, g)$ is a generic pair and by the universal property of $\mathcal{Cl}_{1,0}$ we obtain the algebra homomorphism ϕ such that $g = \phi \circ \pi$. Checking ϕ for bijectivity between basis sets we have

$$\phi(1_{\mathcal{Cl}}) = (1,1)\,, \quad \text{and}$$

$$\phi(e) = \phi\big(\pi(1)\big) = g(1) = (1,-1)\,.$$

Thus, ϕ is an algebra isomorphism.

Proof of 2. Let $\{1_{\mathcal{Cl}}, e_1, e_2, e_{12}\}$ be the standard basis of $\mathcal{Cl}_{2,0}$. First, the isomorphism betweem $\mathcal{Cl}_{2,0}$ and $\mathbb{R}(2)$ is demonstrated. Let $e_1 = \pi(1,0)$ and $e_2 = \pi(0,1)$. Define g on the basis elements of $\mathbb{R}^2$ by

$$g(1,0) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \text{and} \qquad g(0,1) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and extend linearly to all of $\mathbb{R}^2$. Note that

$$g^2(1,0) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \mathrm{Id}\,,$$

$$g^2(0,1) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \mathrm{Id}\,, \text{ and}$$

$$\begin{aligned} g(1,0)g(0,1) &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ &= -1 \cdot \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -1 \cdot g(0,1)g(1,0)\,. \end{aligned}$$

Then, for any $(a,b) \in \mathbb{R}^2$,

$$\begin{aligned} g^2(a,b) &= \big(ag(1,0) + bg(0,1)\big)\big(ag(1,0) + bg(0,1)\big) \\ &= a^2 g^2(1,0) + b^2 g^2(0,1) + abg(1,0)g(0,1) + abg(0,1)g(1,0) \\ &= (a^2 + b^2)\mathrm{Id} = Q(a,b) \cdot \mathrm{Id}\,. \end{aligned}$$

Thus, $\big(\mathbb{R}(2), g\big)$ is a generic pair and so there exists an algebra homomorphism

$\phi : \mathcal{C}\ell_{2,0} \to \mathbb{R}(2)$ such that $g = \phi \circ \pi$. We have

$$\phi(1_{\mathcal{C}\ell}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$\phi(e_1) = \phi\big(\pi(1,0)\big) = g(1,0) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$\phi(e_2) = \phi\big(\pi(0,1)\big) = g(0,1) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \text{ and}$$

$$\phi(e_{12}) = \phi\big(\pi(1,0)\pi(0,1)\big) = g(1,0)g(0,1) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

It is easily shown that these matrices are linearly independent and since both spaces have dimension 4 it follows that ϕ is an algebra isomorphism.

The isomorphism between $\mathcal{C}\ell_{1,1}$ and $\mathbb{R}(2)$ is demonstrated in a similar way. In this case, define $g : \mathbb{R}^2 \to \mathbb{R}(2)$ by specifying

$$g(1,0) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \text{and} \qquad g(0,1) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

and extending linearly to all of $\mathbb{R}^2$. It follows that

$$g^2(1,0) = \mathrm{Id},$$

$$g^2(0,1) = -\mathrm{Id},$$

$$g(1,0)g(0,1) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= -1 \cdot \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = -g(0,1)g(1,0)\,,$$

and so for any $(a,b) \in \mathbb{R}^2$,

$$g^2(a,b) = \big(ag(1,0) + bg(0,1)\big)^2$$

$$= (a^2 - b^2)\mathrm{Id} + abg(1,0)g(0,1) + bag(0,1)g(1,0) =$$

$$= Q(a,b) \cdot \mathrm{Id}\,,$$

and therefore $\big(\mathbb{R}(2), g\big)$ is a generic pair. The unique algebra homomorphism ϕ guaranteed by the universal property of $\mathcal{C}\ell_{1,1}$ is bijective between basis sets:

$$\phi(e_1) = \phi\big(\pi(1,0)\big) = g(1,0)\,,$$

$$\phi(e_2) = \phi\big(\pi(0,1)\big) = g(0,1)\,, \text{ and}$$

$$\phi(e_{12}) = \phi\big(\pi(1,0)\pi(0,1)\big) = g(1,0)g(0,1)\,,$$

and so ϕ is an isomorphism between $\mathcal{C}\ell_{1,1}$ and $\mathbb{R}(2)$. □

6.3 Some Tensor Product Isomorphisms

At this point it is necessary to take a break from our classification and introduce a number of algebra isomorphisms involving tensor products that will be used to classify the remaining Clifford algebras. Certain of these isomorphisms involve matrix algebras whose matrices have entries fom the complex numbers or the quaternions. Despite this, all algebras pertaining to these isomorphisms are real algebras, and likewise, the tensor product spaces are over the field of real numbers. To help make this point apparent, the symbol $\otimes_{\mathbb{R}}$ is used for the tensor product in cases where the base field might otherwise be ambiguous.

In the propositions that follow, the symbol $\mathbb{K}$ will be used to denote $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$ and $\mathbb{K}(n)$ will signify the real algebra of n-by-n matrices with entries from $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$.

The tensor product isomorphisms are summarized here, followed by their proofs.

1. Proposition 6.4: $\mathbb{R}(m) \otimes \mathbb{R}(n) \cong \mathbb{R}(mn)$ for all $m, n \geq 0$.
2. Proposition 6.5: $\mathbb{R}(n) \otimes_{\mathbb{R}} \mathbb{K} \cong \mathbb{K}(n)$ for $\mathbb{K} = \mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$, and all $n \geq 0$.
3. Proposition 6.6: $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C} \oplus \mathbb{C}$.
4. Proposition 6.7: $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{H} \cong \mathbb{C}(2)$.
5. Proposition 6.8: $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{H} \cong \mathbb{R}(4)$.

Proposition 6.4. : $\mathbb{R}(m) \otimes \mathbb{R}(n) \cong \mathbb{R}(mn)$ for all $m, n \geq 0$.

Proof. Let $A = (a_{ij})$ and $A' = (a'_{ij})$ be two arbitrary matrices in $\mathbb{R}(m)$ and B and B' be two arbitrary matrices from $\mathbb{R}(n)$. Define the map $K : \mathbb{R}(m) \times \mathbb{R}(n) \to \mathbb{R}(mn)$ by

$$K(A, B) = \begin{pmatrix} a_{11}B & \cdots & a_{1m}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mm}B \end{pmatrix},$$

where $a_{ij}B$ is an n-by-n block of entries corresponding to the n-by-n matrix $a_{ij}B$. Map K is referred to as the *Kronecker product*.

It follows from the linearity of matrix operations that K is linear in each component, and hence bilinear. By the universal property of the tensor product, $\big(\mathbb{R}(mn), K\big)$ is a generic pair and so there exists a unique linear $L : \mathbb{R}(m) \otimes \mathbb{R}(n) \to \mathbb{R}(mn)$ such that $L(A \otimes B) = K(A, B)$.

From Section 4.3.4, $\mathcal{B} = \{E_{ij} \otimes \widetilde{E}_{pq} \mid 1 \le i, j \le m\,, 1 \le p, q \le n\}$ is a basis for $\mathbb{R}(m) \otimes \mathbb{R}(n)$, where E_{ij} and $\widetilde{E}_{pq}$ are the basis vectors of $\mathbb{R}(m)$ and $\mathbb{R}(n)$, respectively, defined in Example 1.2. Applying L to basis elements in $\mathcal{B}$ yields

$$L(E_{ij} \otimes E_{pq}) = F_{rs}$$

where $r = (i-1)n + p$ and $s = (j-1)n + q$, and F_{rs} is a matrix that contains all zeros except for a 1 in the r^{th} row and s^{th} column. Indices r and s both range from 1 to mn, thus the set of all F_{rs} forms a basis for $\mathbb{R}(mn)$. Having found L to be a linear bijection, it is straightforward to show it is also a homomorphism, which follows from the properties of matrix multiplication. The product AA' has $\sum_k a_{ik}a'_{kj}$ as the entry in the i^{th} row and k^{th} column, therefore

$$
\begin{aligned}
L(AA' \otimes BB') &= \begin{pmatrix} \sum_k a_{1k}a'_{k1}BB' & \cdots & \sum_k a_{1k}a'_{km}BB' \\ \vdots & \ddots & \vdots \\ \sum_k a_{mk}a'_{k1}BB' & \cdots & \sum_k a_{mk}a'_{km}BB' \end{pmatrix} \\
&= \begin{pmatrix} a_{11}B & \cdots & a_{1m}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mm}B \end{pmatrix} \begin{pmatrix} a'_{11}B' & \cdots & a'_{1m}B' \\ \vdots & \ddots & \vdots \\ a'_{m1}B' & \cdots & a'_{mm}B' \end{pmatrix} \\
&= L(A \otimes B)L(A' \otimes B')\,.
\end{aligned}
$$

□

Proposition 6.5. $\mathbb{R}(n) \otimes_{\mathbb{R}} \mathbb{K} \cong \mathbb{K}(n)$ for $\mathbb{K} = \mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$, and all $n \geq 0$.

Proof. As a space of n-by-n matrices, $\mathbb{R}(n)$ has a basis $\mathcal{B} = \{E_{\ell m} \mid 1 \leq \ell, m \leq n\}$ as defined in Example 1.2. The tensor product $\mathbb{R}(n) \otimes \mathbb{C}$ over the field of reals thus has a basis $\{E_{\ell m} \otimes 1, E_{\ell m} \otimes i \mid 1 \leq \ell, m \leq n\}$.

Matrix algebra $\mathbb{C}(n)$ consists of matrices of the form

$$M = \begin{pmatrix} z_{11} & \cdots & z_{1n} \\ \vdots & \ddots & \vdots \\ z_{n1} & \cdots & z_{nn} \end{pmatrix},$$

where $z_{\ell m} = a_{\ell m} + i b_{\ell m}$. Take $\{F_{\ell m} \mid 1 \leq \ell, m \leq n\}$ to be a set of matrices defined analogously to the $E_{\ell m}$ and define $\{F'_{\ell m} \mid 1 \leq \ell, m \leq n\}$ to be the collection of matrices that have elements consisting of all zeros except for an i in the ℓ^{th} row and m^{th} column. Any $M \in \mathbb{C}(n)$ can be expressed as $M = \sum_{\ell,m}(a_{\ell m} F_{\ell m} + b_{\ell m} F'_{\ell m})$ and it is at once evident that $\{F_{\ell m}\} \cup \{F'_{\ell m}\}$ is linearly independent, so the set is a basis for $\mathbb{C}(n)$.

The basis sets for $\mathbb{R}(n) \otimes_{\mathbb{R}} \mathbb{C}$ and $\mathbb{C}(n)$ both have $2n^2$ elements therefore the two spaces are isomorphic as vector spaces. The map L defined by

$$L(E_{\ell m} \otimes 1) = F_{\ell m} \quad \text{and} \quad L(E_{\ell m} \otimes i) = F'_{\ell m}, \tag{6.1}$$

is a bijection between bases that becomes an algebra isomorphism when L is extended linearly. Verification that L is a homomorphism makes use of the following identities

$$E_{\ell m} E_{pq} = \begin{cases} E_{\ell q} & \text{if } p = m, \\ 0 & \text{otherwise,} \end{cases} \qquad \text{(and similarly for } F_{\ell m} F_{pq}\text{),}$$

$$F_{\ell m}F'_{pq} = F'_{\ell m}F_{pq} = \begin{cases} F'_{\ell q} & \text{if } p = m\,, \\ 0 & \text{otherwise,} \end{cases} \qquad \text{and}$$

$$F'_{\ell m}F'_{pq} = \begin{cases} -F_{\ell q} & \text{if } p = m\,, \\ 0 & \text{otherwise,} \end{cases}$$

and is demonstrated, for any $A = \sum_{\ell,m} \alpha_{\ell m} E_{\ell m}$, $B = \sum_{p,q} \beta_{pq} E_{pq} \in \mathbb{R}(n)$ and any $z_r = a_r + ib_r \in \mathbb{C}$, by

$$\begin{aligned}
L\big((A \otimes z_1)(B \otimes z_2)\big) &= L(AB \otimes z_1 z_2) \\
&= L\left(\sum_{\ell,m} \sum_{p,q} \alpha_{\ell m}\beta_{pq} E_{\ell m} E_{pq} \otimes z_1 z_2 \right) \\
&= L\left(\sum_{\ell,q} \sum_{m} \alpha_{\ell m}\beta_{mq} E_{\ell q} \otimes \Big[a_1 a_2 - b_1 b_2 + (a_1 b_2 + a_2 b_1) i \Big] \right) \\
&= a_1 a_2 \sum_{\ell,q} \sum_{m} \alpha_{\ell m}\beta_{mq} F_{\ell q} - b_1 b_2 \sum_{\ell,q} \sum_{m} \alpha_{\ell m}\beta_{mq} F_{\ell q} \\
&\quad + a_1 b_2 \sum_{\ell,q} \sum_{m} \alpha_{\ell m}\beta_{mq} F'_{\ell q} + a_2 b_1 \sum_{\ell,q} \sum_{m} \alpha_{\ell m}\beta_{mq} F'_{\ell q} \\
&= a_1 a_2 \sum_{\ell,m} \sum_{p,q} \alpha_{\ell m}\beta_{pq} F_{\ell m} F_{pq} + b_1 b_2 \sum_{\ell,m} \sum_{p,q} \alpha_{\ell m}\beta_{pq} F'_{\ell m} F'_{pq} \\
&\quad + a_1 b_2 \sum_{\ell,m} \sum_{p,q} \alpha_{\ell m}\beta_{pq} F_{\ell m} F'_{pq} + a_2 b_1 \sum_{\ell,m} \sum_{p,q} \alpha_{\ell m}\beta_{pq} F'_{\ell m} F_{pq} \\
&= \left(a_1 \sum_{\ell,m} \alpha_{\ell m} F_{\ell m} + b_1 \sum_{\ell,m} \alpha_{\ell m} F'_{\ell m} \right) \left(a_2 \sum_{p,q} \beta_{pq} F_{pq} + b_2 \sum_{p,q} \beta_{pq} F'_{pq} \right)
\end{aligned}$$

$$= \left(a_1 L\left(\sum_{\ell,m} \alpha_{\ell m} E_{\ell m} \otimes 1\right) + b_1 L\left(\sum_{\ell,m} \alpha_{\ell m} E_{\ell m} \otimes i\right)\right)$$

$$\cdot \left(a_2 L\left(\sum_{p,q} \beta_{pq} E_{pq} \otimes 1\right) + b_2 L\left(\sum_{p,q} \beta_{pq} E_{pq} \otimes i\right)\right)$$

$$= L\left(\sum_{\ell,m} \alpha_{\ell m} E_{\ell m} \otimes (a_1 + ib_1)\right) L\left(\sum_{p,q} \beta_{pq} E_{pq} \otimes (a_2 + ib_2)\right)$$

$$= L(A \otimes z_1) L(B \otimes z_2)\,.$$

The proof that $\mathbb{R}(n) \otimes_{\mathbb{R}} \mathbb{H} \cong \mathbb{H}(n)$ is similar. A basis for $\mathbb{R}(n) \otimes \mathbb{H}$ is $\{E_{\ell m} \otimes 1, E_{\ell m} \otimes i, E_{\ell m} \otimes j, E_{\ell m} \otimes k \mid 1 \leq \ell, m \leq n\}$ and a basis for $\mathbb{H}(n)$ is $\{F_{\ell m}, F'_{\ell m}, F''_{\ell m}, F'''_{\ell m} \mid 1 \leq \ell, m \leq n\}$ where $F_{\ell m}$ and $F'_{\ell m}$ are defined as before, and $F''_{\ell m}$ and $F'''_{\ell m}$ are defined similarly to $F'_{\ell m}$ except with a j and k, respectively, in the ℓ^{th} row and m^{th} column. The linear map L is defined as in Equation 6.1 with the additional defining equations

$$L(E_{\ell m} \otimes j) = F''_{\ell m} \quad \text{and} \quad L(E_{\ell m} \otimes k) = F'''_{\ell m}\,.$$

Verification that L is a homomorphism follows a similar path as before. □

The following proposition is stated in [LM90] and [Gal09] but the proof is not given. We include the proof here for completeness but the result is not needed subsequently.

Proposition 6.6. $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C} \oplus \mathbb{C}$.

Proof. From Section 4.3.4, the set $\{1 \otimes 1, 1 \otimes i, i \otimes 1, i \otimes i\}$ is a basis for tensor product $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ over the field of reals. From $\mathbb{C} \oplus \mathbb{C}$, the set $\mathcal{B} = \{(1,1), (1,-1), (i,i), (-i,i)\}$ spans the space, since an arbitrary element $(z_1, z_2) = (a_1 + ib_1, a_2 + ib_2)$ can be expressed as

$$\frac{a_1 + a_2}{2}(1,1) + \frac{a_1 - a_2}{2}(1,-1) + \frac{b_1 + b_2}{2}(i,i) + \frac{b_2 - b_1}{2}(-i,i)\,.$$

Since $\mathbb{C}$ has real dimension 2, the direct sum has dimension 2+2 = 4, so these vectors form a basis of $\mathbb{C} \oplus \mathbb{C}$. Define the map L between the basis sets by

$$L(1 \otimes 1) = (1,1)\,, \qquad\qquad L(1 \otimes i) = (i,i)\,,$$
$$L(i \otimes 1) = (-i,i)\,, \qquad \text{and} \qquad L(i \otimes i) = (1,-1)\,,$$

and extend L linearly. We now check that L is a homomorphism. Let $z_p = a_p + ib_p$ and $w_p = c_p + id_p$. Then, for any two elements $z_1 \otimes z_2$ and $w_1 \otimes w_2$ in $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$,

$$\begin{aligned}
L\big((z_1 \otimes w_1)(z_2 \otimes w_2)\big) &= L(z_1w_1 \otimes z_2w_2) \\
&= L\big([a_1c_1 - b_1d_1][a_2c_2 - b_2d_2]1 \otimes 1 + [a_1c_1 - b_1d_1][a_2d_2 + b_2c_2]1 \otimes i \\
&\quad + [a_1d_1 + b_1c_1][a_2c_2 - b_2d_2]i \otimes 1 + [a_1d_1 + b_1c_1][a_2d_2 + b_2c_2]i \otimes i\big) \\
&= [a_1c_1 - b_1d_1][a_2c_2 - b_2d_2] \cdot (1,1) + [a_1c_1 - b_1d_1][a_2d_2 + b_2c_2] \cdot (i,i) \\
&\quad + [a_1d_1 + b_1c_1][a_2c_2 - b_2d_2] \cdot (-i,i) + [a_1d_1 + b_1c_1][a_2d_2 + b_2c_2] \cdot (1,-1)\,.
\end{aligned}$$

Distributing across the scalars followed by factoring and collecting terms gives

$$\begin{aligned}
L(z_1w_1 \otimes z_2w_2) &= \big(a_1a_2(1,1) + a_1b_2(i,i) + a_2b_1(-i,i) + b_1b_2(1,-1)\big) \\
&\quad \cdot \big(c_1c_2(1,1) + c_1d_2(i,i) + c_2d_1(-i,i) + d_1d_2(1,-1)\big) \\
&= \big(L(a_1a_21 \otimes 1) + L(a_1b_21 \otimes i) + L(a_2b_1i \otimes 1) + L(b_1b_2i \otimes i)\big) \\
&\quad \cdot \big(L(c_1c_21 \otimes 1) + L(c_1d_21 \otimes i) + L(c_2d_1i \otimes 1) + L(d_1d_2i \otimes i) \\
&= L\big((a_1 + ib_1) \otimes (a_2 + ib_2)\big)L\big((c_1 + id_1) \otimes (c_2 + id_2)\big) \\
&= L(z_1 \otimes w_1)L(z_2 \otimes w_2)\,.
\end{aligned}$$

□

The proofs of Propositions 6.7 and 6.8 are based on outlines provided in [LM90] and [Gal09].

Proposition 6.7. $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{H} \cong \mathbb{C}(2)$.

Proof. For this proof, the universal property of the tensor product is used to produce a specific algebra isomorphism between the two algebras. We adopt the viewpoint wherein the quaternions are a two-dimensional complex vector space with basis $\{1, j\}$. The set of all complex linear transformations from $\mathbb{H}$ to $\mathbb{H}$ is a vector space which we will denote by $\mathrm{Hom}_{\mathbb{C}}(\mathbb{H}, \mathbb{H})$. The subscript $\mathbb{C}$ serves as a reminder that $\mathbb{H}$ is taken to be a complex vector space. It must be noted that $\mathrm{Hom}_{\mathbb{C}}(\mathbb{H}, \mathbb{H})$ itself is being viewed as a real vector space which is isomorphic to the eight dimensional real vector space $\mathbb{C}(2)$. Function composition on $\mathrm{Hom}_{\mathbb{C}}(\mathbb{H}, \mathbb{H})$ and standard matrix multiplication on $\mathbb{C}(2)$ convert these two spaces into real algebras. As algebras, $\mathrm{Hom}_{\mathbb{C}}(\mathbb{H}, \mathbb{H})$ and $\mathbb{C}(2)$ continue to be isomorphic. This fact will allow us to use both spaces in proving that $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{H}$ and $\mathbb{C}(2)$ are isomorphic.

To begin, define a real bilinear map $\rho' : \mathbb{C} \times \mathbb{H} \to \mathrm{Hom}_{\mathbb{C}}(\mathbb{H}, \mathbb{H})$ by $\rho'(x, z) = \gamma_{xz}$, where $\gamma_{xz}(y) = xy\overline{z}$, for all $x \in \mathbb{C}$ and all $y, z \in \mathbb{H}$. By the universal property of the tensor product, there exists a linear $\rho : \mathbb{C} \otimes_{\mathbb{R}} \mathbb{H} \to \mathrm{Hom}_{\mathbb{C}}(\mathbb{H}, \mathbb{H})$ such that $\rho(x \otimes z) = \rho'(x, z)$ for all $(x, z) \in \mathbb{C} \times \mathbb{H}$. It will now be shown that ρ is in fact a bijective homomorphism, and hence an algebra isomorphism. First, the proof of homomorphism is presented.

For any $u, x \in \mathbb{C}$ and any $w, z \in \mathbb{H}$, we have $\rho(u \otimes w) \circ \rho(x \otimes z) = \gamma_{u,w} \circ \gamma_{x,z}$ and $\rho(ux \otimes wz) = \gamma_{ux,wz}$. For any $y \in \mathbb{H}$,

$$\big(\gamma_{u,w} \circ \gamma_{x,z}\big)(y) = \gamma_{u,w}(xy\overline{z}) = uxy\overline{wz} = \gamma_{ux,wz}(y)\,,$$

It follows that $\rho(u \otimes w) \circ \rho(x \otimes z) = \rho(ux \otimes wz)$ and so ρ is a homomorphism.

Bijectivity of ρ is checked by demonstrating that a basis $\mathcal{B}$ of $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{H}$ maps to a basis of $\mathrm{Hom}_{\mathbb{C}}(\mathbb{H}, \mathbb{H})$. The fact that $\rho(\mathcal{B})$ forms a basis is shown by giving the matrix representation for each linear transformation in $\rho(\mathcal{B})$ and then demonstrating that those matrices form a linearly independent set that spans $\mathbb{C}(2)$.

The matrix representations of $\rho(\mathcal{B})$ are determined from the first eight rows of Table 6.1. The table gives the basis of $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{H}$ and the image of this basis under ρ. The image is a collection of linear transformations; the table summarizes each transformation's values on $\{1, i, j, k\}$. Since $\mathbb{H}$ is regarded as a complex vector space, this means that

$$1 := \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad i := \begin{pmatrix} i \\ 0 \end{pmatrix}, \quad j := \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \text{and} \quad k := \begin{pmatrix} 0 \\ i \end{pmatrix}. \tag{6.2}$$

From Table 6.1, it is deduced that the linear tranformations in $\rho(\mathcal{B})$ are equivalent to the following matrices in $\mathbb{C}(2)$:

$$\gamma_{1,1} := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \gamma_{i,1} := \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix},$$

$$\gamma_{1,i} := \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, \qquad \gamma_{i,i} := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$\gamma_{1,j} := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \gamma_{i,j} := \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix},$$

$$\gamma_{1,k} := \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \qquad \text{and} \qquad \gamma_{i,k} := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Suppose a linear combination of these matrices equaled the zero matrix. The placement of the zeros in each matrix (either in the diagonal or off-diagonal entries), implies that checking for linear independence amounts to checking for linear independence in four sets of two matrices: $\{\gamma_{1,1}\,, \gamma_{i,i}\}$, $\{\gamma_{1,i}\,, \gamma_{i,1}\}$, $\{\gamma_{1,j}\,, \gamma_{i,k}\}$, and $\{\gamma_{1,k}\,, \gamma_{i,j}\}$. It then follows from

Table 6.1: In Propositions 6.7 and 6.8, the universal property of the tensor product is used to produce a linear map ρ. The table summarizes the properties of the linear transformations in $\rho(\mathcal{B})$ and is used in demonstrating that ρ is a bijection.
In Proposition 6.7, $\rho : \mathbb{C} \otimes_{\mathbb{R}} \mathbb{H} \to \mathrm{Hom}_{\mathbb{C}}(\mathbb{H}, \mathbb{H})$ and only the first eight rows of the table are relevant. The quaternion values 1, i, j, and k represent the complex vectors given in Equation 6.2.
In Proposition 6.8, $\rho : \mathbb{H} \otimes_{\mathbb{R}} \mathbb{H} \to \mathrm{Hom}_{\mathbb{R}}(\mathbb{H}, \mathbb{H})$ and all rows of the table are required. In this case, the quaternions are viewed as a four dimensional, real vector space.

tensor product basis element $e_1 \otimes e_2$	$\rho(e_1 \otimes e_2)$	$\gamma_{e_1,e_2}(1)$	$\gamma_{e_1,e_2}(i)$	$\gamma_{e_1,e_2}(j)$	$\gamma_{e_1,e_2}(k)$
$1 \otimes 1$	$\gamma_{1,1}$	1	i	j	k
$1 \otimes i$	$\gamma_{1,i}$	$-i$	1	k	$-j$
$1 \otimes j$	$\gamma_{1,j}$	$-j$	$-k$	1	i
$1 \otimes k$	$\gamma_{1,k}$	$-k$	j	$-i$	1
$i \otimes 1$	$\gamma_{i,1}$	i	-1	k	$-j$
$i \otimes i$	$\gamma_{i,i}$	1	i	$-j$	$-k$
$i \otimes j$	$\gamma_{i,j}$	$-k$	j	i	-1
$i \otimes k$	$\gamma_{i,k}$	j	k	1	i
$j \otimes 1$	$\gamma_{j,1}$	j	$-k$	-1	i
$j \otimes i$	$\gamma_{j,i}$	k	j	i	1
$j \otimes j$	$\gamma_{j,j}$	1	$-i$	j	$-k$
$j \otimes k$	$\gamma_{j,k}$	$-i$	-1	k	j
$k \otimes 1$	$\gamma_{k,1}$	k	j	$-i$	-1
$k \otimes i$	$\gamma_{k,i}$	$-j$	k	-1	i
$k \otimes j$	$\gamma_{k,j}$	i	1	k	j
$k \otimes k$	$\gamma_{k,k}$	1	$-i$	$-j$	k

the placement of the minuses that each of these sets is linearly independent. For example, taking the linear combination $\alpha\gamma_{1,1} + \beta\gamma_{i,i} = 0$ implies that $\alpha + \beta = 0$ and $\alpha - \beta = 0$, that is, that $\alpha = \beta = 0$. So the gammas produced by $\rho(\mathcal{B})$ are linearly independent. Because the dimensions of $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{H}$ and $\mathbb{C}(2)$ are both 8, this means that the gammas span $\mathrm{Hom}_{\mathbb{C}}(\mathbb{H}, \mathbb{H})$, and thus $\rho(\mathcal{B})$ is a basis. □

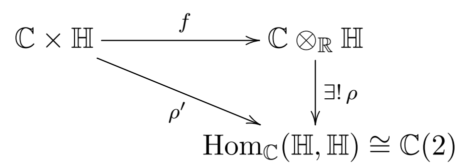

Figure 6.2: A commuting diagram showing the tensor product's universal property as used in Proposition 6.7 to demonstrate the algebra isomorphism between $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{H}$ and $\mathbb{C}(2)$.

Proposition 6.8. $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{H} \cong \mathbb{R}(4)$.

Proof. The proof is similar to Proposition 6.7. In this case, $\mathbb{H}$ is viewed as a four dimensional real vector space. Similar to the previous proof, we use the fact that $\mathrm{Hom}_{\mathbb{R}}(\mathbb{H}, \mathbb{H})$ and $\mathbb{R}(4)$ are algebra isomorphic spaces of dimension 16. Bilinear map $\rho' : \mathbb{H} \times \mathbb{H} \to \mathrm{Hom}_{\mathbb{R}}(\mathbb{H}, \mathbb{H})$ is defined such that $\rho'(x, z) = \gamma_{x,z}$ for any $x, z \in \mathbb{H}$, and $\gamma_{x,z}(y) = xy\bar{z}$. By the universal property of the tensor product, there exists the unique linear map $\rho : \mathbb{H} \otimes_{\mathbb{R}} \mathbb{H} \to \mathrm{Hom}_{\mathbb{R}}(\mathbb{H}, \mathbb{H})$ such that $\rho(x \otimes z) = \rho'(x, z)$ for any $x \otimes z \in \mathbb{H} \otimes_{\mathbb{R}} \mathbb{H}$. Linear map ρ is shown to be a homomorphism in the same manner as Proposition 6.7. That ρ is an algebra ismorphism also follows a similar course to Proposition 6.7, although this time there are twice as many basis vectors that must be checked. Table 6.1 summarizes the values of the gammas on $\{1, i, j, k\}$ from which it is determined that the gammas are represented by the following

4-by-4 real matrices:

$$\gamma_{1,1} := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \gamma_{j,1} := \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix},$$

$$\gamma_{1,i} := \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad \gamma_{j,i} := \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

$$\gamma_{1,j} := \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \qquad \gamma_{j,j} := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

$$\gamma_{1,k} := \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \qquad \gamma_{j,k} := \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

$$\gamma_{i,1} := \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad \gamma_{k,1} := \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

$$\gamma_{i,i} := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \gamma_{k,i} := \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

$$\gamma_{i,j} := \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad \gamma_{k,j} := \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

$$\gamma_{i,k} := \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad \gamma_{k,k} := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Suppose a linear combination of the gammas equals the zero matrix. Because of the placement of the zeros in the individual gammas, checking for linear independence of the sixteen matrices amounts to checking for linear independence in four sets of four matrices, namely, $\{\gamma_{1,1}, \gamma_{i,i}, \gamma_{j,j}, \gamma_{k,k}\}$, $\{\gamma_{1,i}, \gamma_{i,1}, \gamma_{j,k}, \gamma_{k,j}\}$, $\{\gamma_{1,j}, \gamma_{j,1}, \gamma_{i,k}, \gamma_{k,i}\}$, and $\{\gamma_{1,k}, \gamma_{k,1}, \gamma_{i,j}, \gamma_{j,i}\}$.

The linear independence of any one of these sets is equivalent to the linear independence of the following vectors up to a factor of -1:

$$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}.$$

The 4-by-4 matrix having these vectors as its columns is non-singular, so the vectors are linearly independent, and hence so are the gamma matrices.

□

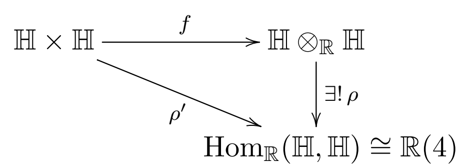

Figure 6.3: A commuting diagram showing the tensor product's universal property as used in Proposition 6.8 to demonstrate the algebra isomorphism between $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{H}$ and $\mathbb{R}(4)$.

6.4 Tensor Product Decompositions of Clifford Algebras

The tensor product isomorphisms in this section allow one to decompose each of the remaining Clifford algebras into a tensor product of the lowest dimensional Clifford algebras whose isomorphic matrix algebras were already found in Sections 6.1 and 6.2. With the three tensor product decompositions in Theorem 6.9 and the isomorphisms already presented in this chapter, all real Clifford algebras can be classified. In order for the decompositions to hold in the $n = 0$ and $p = q = 0$ cases, we define $\mathcal{C}\ell_{0,0} := \mathbb{R}$.

Theorem 6.9. We have the following isomorphisms

1. $\mathcal{C}\ell_{0,n+2} \cong \mathcal{C}\ell_{n,0} \otimes \mathcal{C}\ell_{0,2}$,

2. $\mathcal{C}\ell_{n+2,0} \cong \mathcal{C}\ell_{0,n} \otimes \mathcal{C}\ell_{2,0}$, and

3. $\mathcal{C}\ell_{p+1,q+1} \cong \mathcal{C}\ell_{p,q} \otimes \mathcal{C}\ell_{1,1}$,

for all $n, p, q \geq 0$ and $\mathcal{C}\ell_{0,0} := \mathbb{R}$.

Proof. The proofs for these isomorphisms all follow the same general prescription, which is that of [LM90] and [Gal09]. In each case, a generic function (in the sense of the universal property for the Clifford algebra) is found from the underlying vector space ($\mathbb{R}^{n+2}$ in parts 1 and 2, $\mathbb{R}^{p+q+2}$ in part 3) of the Clifford algebra on the left-hand side of the equation to the tensor product space on the right-hand side. The Clifford algebra's universal property then guarantee's the homomorphism between the two algebras and this homomorphism is shown to be bijective.

Proof of 1. Let $\{e_1, \ldots, e_{n+2}\}$ be a basis for $\mathbb{R}^{n+2}$ which is orthonormal with respect to the standard inner product. Let $\{e'_1, \ldots, e'_n\}$ be generators for $\mathcal{C}\ell_{n,0}$ and $\{e''_1, e''_2\}$ be generators for $\mathcal{C}\ell_{0,2}$. Let $f : \mathbb{R}^{n+2} \to \mathcal{C}\ell_{n,0} \otimes \mathcal{C}\ell_{0,2}$ be a map defined on the given basis by

$$f(e_i) = \begin{cases} e'_i \otimes e''_1 e''_2 & \text{for } 1 \leq i \leq n \\ 1 \otimes e''_{i-n} & \text{for } n+1 \leq i \leq n+2\,, \end{cases}$$

and extended linearly to all of $\mathbb{R}^{n+2}$. In order for f to be a generic function of the $\mathcal{C}\ell_{0,n+2}$ universal property, it must be the case that $f(v)^2 = Q_{0,n+2}(v)(1 \otimes 1) = -\|v\|^2 (1 \otimes 1)$ for all $v \in \mathbb{R}^{n+2}$. This is indeed the case as we now show. Consider v as a linear combination of basis elements $v = \sum_{i=1}^{n+2} \alpha_i e_i$. Then

$$\begin{aligned} f(v)^2 &= \left(\sum_{i=1}^{n+2} \alpha_i f(e_i) \right) \left(\sum_{j=1}^{n+2} \alpha_j f(e_j) \right) \\ &= \sum_{i=1}^{n+2} \alpha_i^2 f(e_i)^2 + \sum_{\substack{i=1 \\ j>i}}^{n+1} \alpha_i \alpha_j \left[f(e_i) f(e_j) + f(e_j) f(e_i) \right]. \end{aligned} \tag{6.3}$$

In computing the right-hand side of Equation 6.3, there are a few cases to consider depending on the values of indices i and j. Looking at the first term, we have

$$f(e_i)^2 = \begin{cases} (e'_i)^2 \otimes (e''_1 e''_2)^2 & \text{for } 1 \leq i \leq n \\ 1 \otimes (e''_{i-n})^2 & \text{for } n+1 \leq i \leq n+2\,. \end{cases}$$

Noting that $(e'_i)^2 = 1$, $(e''_1)^2 = (e''_2)^2 = -1$, and $e''_1 e''_2 = -e''_2 e''_1$ we have that $f(e_i)^2 = -1 \otimes 1$ for any i. The second term contains

$$f(e_i)f(e_j)+f(e_j)f(e_i) = \begin{cases} (e'_i e'_j + e'_j e'_i) \otimes (e''_1 e''_2)^2 & \text{for } 1 \leq i, j \leq n \\ e'_i \otimes (e''_1 e''_2 e''_{j-n} + e''_{j-n} e''_1 e''_2) & \text{for } 1 \leq i \leq n \text{ and } n+1 \leq j \leq n+2 \\ 1 \otimes (e''_{i-n} e''_{j-n} + e''_{j-n} e''_{i-n}) & \text{for } n+1 \leq i, j \leq n+2\,. \end{cases}$$

In each case, the expression is equal to zero. In the first case it is because $e'_i e'_j = -e'_j e'_i$. In the second case it is because $(j-n)$ equals 1 or 2, and in the third case, since $(i-n,\, j-n) = (1,2)$ or $(i-n,\, j-n) = (2,1)$. Thus, Equation 6.3 becomes

$$f(v)^2 = -\sum_{i=1}^{n+2} \alpha_i^2\, 1 \otimes 1 = -\left\|v\right\|^2 1 \otimes 1\,.$$

So we have an algebra homomorphism $\tilde{f}$ between $\mathcal{C}\ell_{n,0} \otimes \mathcal{C}\ell_{0,2}$ and $\mathcal{C}\ell_{0,n+2}$. Since $\tilde{f}$ is a surjective map from the generators of its domain to the generators of its target space, it is also surjective between the entire domain and target space.

Proof of 2. The second isomorphism is proved similarly.

Proof of 3. Let $\{e_1, \ldots, e_{p+1}, \epsilon_1, \ldots, \epsilon_{q+1}\}$ be an orthonormal basis for $\mathbb{R}^{p+q+2}$. Couple this with a quadratic form, $Q_{p+1,q+1}$, such that $Q_{p+1,q+1}(e_i) = 1$ for $1 \leq i \leq p+1$ and $Q_{p+1,q+1}(\epsilon_j) = -1$ for $1 \leq j \leq q+1$. Let $\{e'_1, \ldots, e'_p, \epsilon'_1, \ldots, \epsilon'_q\}$ be a set of generators for $\mathcal{C}\ell_{p,q}$ and $\{e''_1, \epsilon''_1\}$ be generators for $\mathcal{C}\ell_{1,1}$. Now let $f : \mathbb{R}^{p+q+2} \to \mathcal{C}\ell_{p,q} \otimes \mathcal{C}\ell_{1,1}$ be a linear map defined by

$$f(e_i) = \begin{cases} e'_i \otimes e''_1\epsilon''_1 & \text{for } 1 \le i \le p \\ 1 \otimes e''_1 & \text{for } i = p+1\,, \end{cases} \qquad \text{and} \qquad f(\epsilon_j) = \begin{cases} \epsilon'_j \otimes e''_1\epsilon''_1 & \text{for } 1 \le j \le q \\ 1 \otimes \epsilon''_1 & \text{for } j = q+1\,. \end{cases}$$

As with the proof of (1), it is necessary to show that f is a generic function, i.e., that $f(x)^2 = Q_{p+1,q+1}(x)\, 1 \otimes 1$. The remainder of the proof follows a similar argument as for the proof of part 1. □

6.5 Periodicity of 8

The tensor product decompositions of the previous section reveal a certain property of the relationships between the Clifford algebras themselves. This relationship, called the periodicity of 8, was first discovered by Élie Cartan in 1908 [Lou01] and later independently discovered again by Raoul Bott [Gal09].

Theorem 6.10 (Cartan-Bott Periodicity of 8 Theorem)**.** For all $n \ge 0$, we have the following isomorphisms

$$\mathcal{C}\ell_{0,n+8} \cong \mathcal{C}\ell_{0,n} \otimes \mathcal{C}\ell_{0,8}\,, \quad \text{and}$$

$$\mathcal{C}\ell_{n+8,0} \cong \mathcal{C}\ell_{n,0} \otimes \mathcal{C}\ell_{8,0}\,.$$

Proof. Starting with $\mathcal{C}\ell_{0,n+8}$ and applying part 1 of Theorem 6.9 we obtain a string of isomorphisms,

$$\begin{aligned}
\mathcal{C}\ell_{0,n+8} &\cong \mathcal{C}\ell_{n+6,0} \otimes \mathcal{C}\ell_{0,2} \\
&\cong \mathcal{C}\ell_{0,n+4} \otimes \mathcal{C}\ell_{2,0} \otimes \mathcal{C}\ell_{0,2} \\
&\cong \mathcal{C}\ell_{n+2,0} \otimes \mathcal{C}\ell_{0,2} \otimes \mathcal{C}\ell_{2,0} \otimes \mathcal{C}\ell_{0,2} \\
&\cong \mathcal{C}\ell_{0,n} \otimes \mathcal{C}\ell_{2,0} \otimes \mathcal{C}\ell_{0,2} \otimes \mathcal{C}\ell_{2,0} \otimes \mathcal{C}\ell_{0,2} \qquad (6.4) \\
&\cong \mathcal{C}\ell_{0,n} \otimes \mathcal{C}\ell_{0,4} \otimes \mathcal{C}\ell_{2,0} \otimes \mathcal{C}\ell_{0,2} \\
&\cong \mathcal{C}\ell_{0,n} \otimes \mathcal{C}\ell_{6,0} \otimes \mathcal{C}\ell_{0,2} \\
&\cong \mathcal{C}\ell_{0,n} \otimes \mathcal{C}\ell_{0,8} \,.
\end{aligned}$$

The second isomorphism of the theorem is demonstrated in a similar manner using part 2 of Theorem 6.9. □

The next corollary follows readily from our derivation of the Cartan-Bott periodicity theorem.

Corollary 6.11. The following Clifford algebras are isomorphic to the algebra of real 16-by-16 matrices:

$$\mathcal{C}\ell_{0,8} \cong \mathcal{C}\ell_{8,0} \cong \mathbb{R}(16)\,.$$

Proof. In Theorem 6.10, starting with the right-hand side of Equation 6.4 and working down, what is essentially shown is that

$$\mathcal{C}\ell_{2,0} \otimes \mathcal{C}\ell_{0,2} \otimes \mathcal{C}\ell_{2,0} \otimes \mathcal{C}\ell_{0,2} \cong \mathcal{C}\ell_{0,8}\,.$$

Using the isomorphisms $\mathcal{C}\ell_{0,2} \cong \mathbb{H}$ (Proposition 6.2) and $\mathcal{C}\ell_{2,0} \cong \mathbb{R}(2)$ (Proposition 6.3), we have that

$$
\begin{aligned}
\mathcal{C}\ell_{2,0} \otimes \mathcal{C}\ell_{0,2} \otimes \mathcal{C}\ell_{2,0} \otimes \mathcal{C}\ell_{0,2} &\cong \mathbb{R}(2) \otimes \mathbb{H} \otimes \mathbb{R}(2) \otimes \mathbb{H} \\
&\cong \mathbb{R}(2) \otimes \mathbb{R}(2) \otimes \mathbb{H} \otimes \mathbb{H} \quad \text{(Proposition 4.1)} \\
&\cong \mathbb{R}(4) \otimes \mathbb{R}(4) \cong \mathbb{R}(16) \quad \text{(Propositions 6.4 and 6.8).}
\end{aligned}
$$

The proof that $\mathcal{C}\ell_{8,0} \cong \mathbb{R}(16)$ is similar. □

6.6 Summary of the Classification

So far in this chapter we have amassed a bevy of isomorphisms. These can now be used to classify the Clifford algebras by showing that each Clifford algebra is isomorphic to a matrix algebra, or a two-fold direct sum of a matrix algebra with itself. The matrix algebras consist of matrices which take entries from $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$, however, it must be kept in mind that the algebras are in fact real algebras.

The Clifford algebras associated with vector spaces of dimension one or two were shown to be isomorphic to $\mathbb{C}$, $\mathbb{H}$, $\mathbb{R} \oplus \mathbb{R}$, or $\mathbb{R}(2)$. Other Clifford algebras can be decomposed into tensor products of these. Using the isomorphisms of Section 6.3, each tensor product then collapses to the matrix algebra, or matrix algebra direct sum, that classifies the Clifford algebra.

The classification up to $\mathcal{C}\ell_{8,8}$ is given in Table 6.2, which is adapted from [LM90]. The row $r = 0$ and column $c = 0$ of the table can be deduced using Propositions 6.1–6.5, 6.7–6.8, and Theorem 6.9, parts 1 and 2. The remaining entries are obtained using, in addition, part 3 of Theorem 6.9.

Table 6.2: Clifford algebra $\mathcal{C}\ell_{r,c}$ is isomorphic to the matrix algebra in row r and column c of the table. The table is adapted from [LM90].

8	$\mathbb{R}(16)$	$\mathbb{C}(16)$	$\mathbb{H}(16)$	$\mathbb{H}(16) \oplus \mathbb{H}(16)$	$\mathbb{H}(32)$	$\mathbb{C}(64)$	$\mathbb{R}(128)$	$\mathbb{R}(128) \oplus \mathbb{R}(128)$	$\mathbb{R}(256)$
7	$\mathbb{C}(8)$	$\mathbb{H}(8)$	$\mathbb{H}(8) \oplus \mathbb{H}(8)$	$\mathbb{H}(16)$	$\mathbb{C}(32)$	$\mathbb{R}(64)$	$\mathbb{R}(64) \oplus \mathbb{R}(64)$	$\mathbb{R}(128)$	$\mathbb{C}(128)$
6	$\mathbb{H}(4)$	$\mathbb{H}(4) \oplus \mathbb{H}(4)$	$\mathbb{H}(8)$	$\mathbb{C}(16)$	$\mathbb{R}(32)$	$\mathbb{R}(32) \oplus \mathbb{R}(32)$	$\mathbb{R}(64)$	$\mathbb{C}(64)$	$\mathbb{H}(64)$
5	$\mathbb{H}(2) \oplus \mathbb{H}(2)$	$\mathbb{H}(4)$	$\mathbb{C}(8)$	$\mathbb{R}(16)$	$\mathbb{R}(16) \oplus \mathbb{R}(16)$	$\mathbb{R}(32)$	$\mathbb{C}(32)$	$\mathbb{H}(32)$	$\mathbb{H}(32) \oplus \mathbb{H}(32)$
4	$\mathbb{H}(2)$	$\mathbb{C}(4)$	$\mathbb{R}(8)$	$\mathbb{R}(8) \oplus \mathbb{R}(8)$	$\mathbb{R}(16)$	$\mathbb{C}(16)$	$\mathbb{H}(16)$	$\mathbb{H}(16) \oplus \mathbb{H}(16)$	$\mathbb{H}(32)$
3	$\mathbb{C}(2)$	$\mathbb{R}(4)$	$\mathbb{R}(4) \oplus \mathbb{R}(4)$	$\mathbb{R}(8)$	$\mathbb{C}(8)$	$\mathbb{H}(8)$	$\mathbb{H}(8) \oplus \mathbb{H}(8)$	$\mathbb{H}(16)$	$\mathbb{C}(32)$
2	$\mathbb{R}(2)$	$\mathbb{R}(2) \oplus \mathbb{R}(2)$	$\mathbb{R}(4)$	$\mathbb{C}(4)$	$\mathbb{H}(4)$	$\mathbb{H}(4) \oplus \mathbb{H}(4)$	$\mathbb{H}(8)$	$\mathbb{C}(16)$	$\mathbb{R}(32)$
1	$\mathbb{R} \oplus \mathbb{R}$	$\mathbb{R}(2)$	$\mathbb{C}(2)$	$\mathbb{H}(2)$	$\mathbb{H}(2) \oplus \mathbb{H}(2)$	$\mathbb{H}(4)$	$\mathbb{C}(8)$	$\mathbb{R}(16)$	$\mathbb{R}(16) \oplus \mathbb{R}(16)$
0	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{H} \oplus \mathbb{H}$	$\mathbb{H}(2)$	$\mathbb{C}(4)$	$\mathbb{R}(8)$	$\mathbb{R}(8) \oplus \mathbb{R}(8)$	$\mathbb{R}(16)$
	0	1	2	3	4	5	6	7	8

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Curriculum Vitae

Mr. Neilson graduated from The College of William & Mary with a bachelor's degree, double-majoring in Physics and Biology. Afterwards, he earned a master's degree in Physics from the University of Florida. As of 2012, he has worked for nine years at Science Applications International Corporation as an engineer, first in the aerospace field and subsequently in signal processing. Mr. Neilson graduated from George Mason University with a master's degree in Mathematics in the summer of 2012.